CSC 2420F 2010, Assignment 2
Due: November 10 at start of class

1. This problem concerns the knapsack problem. The input is a set of n items $\{(s_1, v_1), \ldots (s_n, v_n)\}$ and a knapsack capacity bound C. Here s_i is the size of item i and v_i is the value of that item. The goal is to find a "feasible solution" so as to maximize the profit of the solution. Without loss of generality let us assume that the size s_i of every item is at most the capacity C of the knapsack so that any single item is a feasible solution.

   (a) In the fractional knapsack problem, we are allowed to place a fractional amount f_i of an item (s_i, v_i) in the knapsack; that is a feasible solution for input $\{(s_1, v_1), (s_2, v_2) \ldots, (s_n, v_n); C\}$ is a vector $(f_1, f_2 \ldots, f_n)$ such that $\sum_{i=1}^{n}(f_i \cdot s_i) \leq C$ and $0 \leq f_i \leq 1$ for all i and the profit of such a solution is $\sum_{i=1}^{n}(f_i \cdot v_i)$. Design a simple optimal greedy algorithm for this problem and argue why your algorithm is optimal.

   (b) In the standard (i.e. $\{0, 1\}$) NP-hard knapsack problem, a feasible solution is a subset $S \subseteq \{1, 2, \ldots, n\} : \sum_{i \in S} s_i \leq C$ and the profit of such a solution is $\sum_{i \in S} v_i$. There are (at least) two natural greedy algorithms for the standard {0,1} knapsack problem; namely, we can sort so that $v_1 \geq v_2 \ldots \geq v_n$ or sort so that $v_1/s_1 \geq v_2/s_2 \ldots \geq v_n/s_n$ and then accept items greedily. Show that both of these greedy algorithms are not c-approximation algorithms for any constant c.

   (c) Lets call the first (resp. second) greedy algorithm as above G_1 (resp. G_2). Show that if we take the best solution obtained by G_1 or G_2 we obtain a 2-approximation.

   (d) Show that no priority algorithm can achieve a constant approximation for the standard knapsack problem.

   (e) (Bonus)

   What is the best inapproximation result for the standard knapsack problem when we restrict algorithms to compute the best solution amongst two priority algorithms.

2. We wish to define an independence system that generalizes both $k+1$ clawfree graphs and matroids. Let us define a k-exchangeable independence system $M = (E, \mathcal{F})$ to be one satisfying the following property (beyond the hereditary property): Let $C \subseteq D \in \mathcal{F}$ (i.e. C and D are independent sets) and $C + x \in \mathcal{F}$. Then $\exists Y \subseteq D - C$ with $|Y| \leq k$ such that $D - Y + x \in \mathcal{F}$.

   (a) Show that M is a matroid if and only if M is a 1-exchangeable independence system.

   (b) Show that independence in a k+1 clawfree graph is a k-exchangeable independence system.

   (c) Show that the following standard greedy algorithm obtains a k-approximation for weighted MIS in a k-exchangeable independence system.

   $S := \emptyset$
   Sort elements in E so that $w_1 \geq w_2 \ldots \geq w_n$
   For $i : 1..n$
       If $S + e_i \in \mathcal{F}$ then $S := S + e_i$
   End For

   Hint: First establish the following lemma:
   Lemma: If $M = (E, \mathcal{F})$ is a k-exchangeable system and Greedy chooses e_i as its i^{th} element, then $w(OPT(S_i)) \leq w(OPT(S_{i+1})) + (k-1)w_{i+1}$. Here $OPT(S)$ denotes a most profitable extension of an independent set S, $S_i = S_{i-1} + e_i$ and w_j is the weight of e_j.

3. (Bonus questions)

   (a) Consider the proportional profit interval selection problem. Suppose we represent input items (ie intervals) by $I_j = (s_j, f_j, w_j, N_j)$ where $N_j = \{k : I_k \cap I_j \neq \emptyset\}$. What is the best priority inapproximation you can prove for the proportional profit interval selection problem in this input model? You might try first the less informative input model where $I_j = (s_j, f_j, w_j, d_j)$ where d_j is the graph theoretic degree of I_j

(b) I have posted a recent paper that we have been working on. The paper concerns the sum coloring (SC) and sum multi-coloring problems (SMC). More specifically, we were considering these problems with regard to $k+1$-clawfree graphs and very specifically, we were considering the problem with respect to proper interval graphs (which are 3-clawfree). As mentioned in class, the (preemptive) SMC problem has a simple (fixed order priority) greedy algorithm that attains a 2-approximation for 3-clawfree graphs whereas the inapproximation bounds we have for various priority models are relatively weak. Can you improve these inapproximations and/or (better yet) improve the 2-approximation for SMC on proper interval graphs by a greedy algorithm (or any efficient algorithm)?

4. This question concerns local search for the exact Max-2-Sat problem where the input is a CNF formula with exactly 2 literals per clause. The goal is to maximize the number (or the total weight) of clauses that can be satisfied by some truth assignment. Khanna et al (link to paper is on web site) consider the locality gap achieved by oblivious and non-oblivious local search. (See pages 14-17 for Max-2-Sat and in particular, the proof of the $\frac{2}{3}$ locality gap for the 1-flip neighbourhood oblivious search starts at the bottom of page 16.) As suggested by Khanna et al, one can also obtain a locaity gap of $\frac{3}{4}$ by an oblivious local search algorithm that defines the neighbourhood of a solution (i.e. truth assignment) to include flipping any single variable and also flipping all variables. Modify the $\frac{2}{3}$ locality gap result to obtain the improved locality gap for this larger neighbourhood.