CSC2420: Algorithm Design, Analysis and Theory

Scribe: Recep Colak **Lecturer:** Allan Borodin
Topic: Linear Programming and Duality **Date:** 2010/10/06

4.1 Introduction

In the first three lectures we have reviewed the fundamental algorithmic approaches including Greedy, Brute Force , Dynamic Programming and Local Search algorithms. At today's lecture, we will cover Linear Programming (LP) duality, which is a widely used class of algorithms for optimization problems. LP covers a wide variety of problems for which both the objective function and the constraints can be represented as linear functions. LP is also interesting from the historical development point of view. *Simplex Method* , which was developed by George Dantzig in 1947, was the first method developed for LP and is still a widely used algorithm for solving LP problems. However, Simplex (for known oivot rules) is not a polynomial algorithm and it is possible to come up with cases where it takes exponential time. The *Ellipsoid Algorithm* (Khachiyan, 1979) on the other hand is a polynomial algorithm in theory but in practice it usually performs poorly compared to *Simplex*. Later in 1984, Narenda Karmarkar came up with the *Interrior Point Method*, which is both provably polynomial and works well in practice.

4.1.1 Recommended Reading

- The Primal-Dual Method for Approximation Algorithms by David P. Williamson
- Lecture notes from Shuchi Chawla's course

4.2 Duality

Since, we briefly went over Linear Programming (LP) and Integer Programming (IP) in the context of set cover, graph orientation and makespan problems, we will start with Duality.

Assume we have a minimization LP problem in canonical from.

$$\begin{aligned} \min \quad & \sum_{j \in \{1,\ldots,n\}} x_j c_j \\ \text{s.t.} \quad & \sum_{j \in \{1,\ldots,n\}} a_{ij} x_j \geq b_j \quad \text{for all } i \in \{1,\ldots,m\} \\ & x_j \geq 0 \end{aligned}$$

We call this the primal (P) in the canonical form. The matrix-vector notation is as follows:

$$\begin{array}{ll} \min & \vec{c}^T\vec{x} \\ \text{s.t.} & A\vec{x} \geq \vec{b} \\ & \vec{x} \geq \vec{0} \end{array}$$

The dual (D) of the primal is given as follows:

$$\begin{array}{ll} \max & \vec{b}^T\vec{y} \\ \text{s.t.} & A^T\vec{y} \leq \vec{c} \\ & \vec{y} \geq \vec{0} \end{array}$$

which corressponds to :

$$\begin{array}{lll} \max & \sum_{i\in\{1,\ldots,m\}} y_i b_i & \\ \text{s.t.} & \sum_{i\in\{1,\ldots,m\}} a_{ij} y_i \leq c_j & \text{for all } j \in \{1,\ldots,n\} \\ & y_i \geq 0 & \end{array}$$

The primal problem and the dual problem are complementary. A finite optimal value to either one determines an optimal value to both. The relation between these two can sometimes be easy to interpret for example in the case of cover vs packing, max flow vs min-cut, etc. However, the interpretation of the dual may not always be intuitively meaningful. Still, duality is very useful because the duality principle states that optimization problems may be viewed from either of two perspectives and this might be useful as the solution of the dual might be much cheaper to calculate than the solution of the primal. Moreover, the relation between P and D will give us the Primal-Dual algorithm, in which we start with feasible solutions x and y and iteratively change the values to satisfy the conditions more and more until we hit the optimal solutions.

4.3 Examples

4.3.1 Set Cover

Given $S = \{S_1, S2,..., S_m\}$ such that $S_i \in E$, $E = \{e_1, e_2, ..., e_n\}$ and w_j is the weight of S_j, we would like to find $S' \subseteq S$ s.t. $\cup_{S_j \in S'} = E$ and $\sum_{S_j \in S'} w_j$ is minimum.

Note that, Vertex Cover is a special case of Set Cover because each vertex v becomes a set S_j that contains v_j's edges. Every universe element, i.e. edge, occurs in exactly 2 sets. Hence, vertex cover is a 2-Frequency set cover problem.

More generally, in an f-Frequency set cover problem, no universe element can occur in more than f sets. For d-Degree (or d-Cardinality) set cover problem, i.e. $|S_j| \leq d$, it is possible to get an H_d approximation, where:

$$H_d = 1 + \frac{1}{2} + \ldots + \frac{1}{d} \approx \ln d$$

Next we define the primal and the dual of the set cover problem.

(P)

$$\begin{array}{rl} \min & \sum_j^m w_j x_j \\ \text{s.t.} & \sum_{j: e_j \in S_j} x_j \geq 1 \quad i \in \{1, \ldots, n\} \\ (IP) & x_j \in \{0, 1\} \\ (LP) & 1 \geq x_j \geq 0 \end{array}$$

(D)

$$\begin{array}{rl} \max & \sum_i^n y_i \\ \text{s.t.} & \sum_{i: e_i \in S_j} y_i \leq w_j \\ & y_i \geq 0 \end{array}$$

4.3.2 Toy example

(P)

$$\begin{array}{rl} \min & x + 4z \\ \text{s.t.} & x + 2z \geq 5 \\ & 2x + z \geq 4 \\ & x, z \geq 0 \end{array}$$

As a starting point:

$$\begin{array}{l} x + 4z \geq x + 2x \geq 4 \\ x + 4z \geq \frac{1}{2}(x + 2z) + \frac{1}{4}(2x + z) \\ x + 4z \geq 2.5 + 1 \\ x + 4z \geq 3.5 \end{array}$$

In general,

$$\begin{array}{l} (x + 2z)u \geq 5u \\ (2x + z)v \geq 4v \\ u, v \geq 0 \end{array}$$

We can now define the dual.

(D)

$$
\begin{array}{ll}
\max & 5u + 4v \\
\text{s.t.} & u + 2v \leq 1 \\
 & 2u + v \leq 4 \\
 & u, v \geq 0
\end{array}
$$

4.4 Weak and Strong Duality

Theorem 4.4.1 (Weak Duality) *If x and y are primal and resp. dual solutions, then $D(y) \leq P(x)$.*

Proof:

All we need to do is to show that $D(y) = b^t y \leq c^t x = P(x)$

$$
\begin{aligned}
b^T y &= \sum_{j=1}^{m} b_j y_j \\
&\leq \sum_{j=1}^{m} (\sum_{i=1}^{n} A_{ji} x_i) y_j \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{m} (A_{ji} j_j) x_i \\
&\leq \sum_{i=1}^{n} c_i x_i \\
&\leq c^T x
\end{aligned}
$$

■

Theorem 4.4.2 (Strong Duality) *If x^* and y^* are (finite) optimal primal and resp. dual solutions, then $D(y^*) = P(x^*)$.*

Proof: Omitted. ■

Corollary 4.4.3 (Complementary Slackness) *Let x and y be finite feasible solutions to (P) and resp. (D). Then x and y are optimal iff the following hold*

- $\forall i \ (b_i - \sum_j A_{ij} x_j)) y_i = 0$
- $\forall j \ (\sum_i A_{ij} i_i) - c_j) x_j = 0$

Proof: Note that the fact that x and y are feasible implies that $(b_i - \sum_j A_{ij} x_j)) y_i \geq 0$ and $(\sum_i A_{ij} i_i) - c_j) x_j \geq 0$. If we sum these over all i and j, we get:

$$
\begin{aligned}
&\sum_i b_i y_i - \sum_{i,j} A_{ij} x_j y_i + \sum_{i,j} A_{ij} y_i x_j - \sum_j c_j x_j \\
&= \sum_i b_i y_i - \sum_j c_j x_j \\
&= 0
\end{aligned}
$$

As such, the inequalites must indeed be equalities.

■

This corollary implies that feasible solutions that satisfy complementary slackness conditions are optimal. As such, for a given problem we can start with feasible solutions x and y and iteratively change the values to satisfy the conditions more and more. This is the core of the Primal-Dual algorithm.

Note that the primal dual method avoids solving the LP and is usually much more efficient that LP rounding.

Now we can give the f-Approximation algorithm for the f-Frequency set cover problem.

procedure *Primal_Dual_Set_Cover*(S, E, W)

while (P) is not feasible, i.e. $\exists e_k$ not covered **do**
    Raise y_k so that some dual constraint (containing y_k) say j^{th} becomes tight
    Freeze all other y_i in j^{th} constraint
    $x_j = 1$, i.e. place S_j in the cover and therefore cover x_k
end while

The above algorithm works becaue if e_k is not covered, that means sets containing e_k are not covered yet. This in turn means the corresponding y_k must have some slack, i.e. y_k can be increased without violating constraints.

Next, we give the Greedy algorithm for the set cover. The core idea is that at each iteration we select the set that minimizes the cost per new additional item to be covered.

procedure *Greedy_Set_Cover*(S, E, W)

$I = \emptyset$
$\hat{S}_j = S_j \ \forall j$
while $\exists e_k$ that is not covered **do**
    $l = argmin_{j:\hat{S}_j \neq \emptyset} \frac{w_j}{|\hat{S}_j|}$
    $I = I \cup \{l\}$
    $\hat{S}_j = \hat{S}_j - S_l \ \forall j$
end while