CSC2401 Introduction to Complexity Theory
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Lecture 15

Polynomial Time Hierarchy

Basic definitions (review)

In the previous lecture, complexity classes Σ_k^P, Δ_k^P, Π_k^P, and PH were defined as follows:

- $\Sigma_0^P = \Pi_0^P = \Delta_0^P = P$;
- $\Sigma_{k+1}^P = NP^{\Sigma_k^P} = \{L \mid L = \mathcal{L}(M^B)\}$, where M^B is a non-deterministic polynomial time bounded Turing machine with some oracle $B \in \Sigma_k^P$;
- $\Delta_{k+1}^P = P^{\Sigma_k^P}$;
- $\Pi_k^P = co\,\Sigma_k^P$;
- $PH = \bigcup\limits_{k \geq 0} \Sigma_k^P$.

Some simple facts:

1) $\Sigma_1^P = NP$ because $\Sigma_1^P = NP^{\Sigma_0^P} = NP^P = NP$;
$\Pi_1^P = coNP$;
$\Delta_1^P = P$, because $\Delta_1^P = P^P = P$.

2) $A \in NP^B \Longrightarrow A \in NP^{\overline{B}}$.

3) $NP^{PSPACE} = PSPACE = P^{PSPACE}$.

Main theorem

$$\Sigma_k^P \cup \Pi_k^P \subseteq \Delta_{k+1}^P \subseteq \Sigma_{k+1}^P \cap \Pi_{k+1}^P \subseteq PSPACE$$

Proof

i) Since $P^A = P^{\overline{A}}$, $\Pi_k^P \subseteq P^{\Pi_k^P} = P^{\Sigma_k^P} = \Delta_{k+1}^P$.
Also, $\Sigma_k^P = co\ \Pi_k^P \subseteq \overline{\Delta}_{k+1}^P = \Delta_{k+1}^P$.
Therefore, $\Sigma_k^P \cup \Pi_k^P \subseteq \Delta_{k+1}^P$.

ii) $\Delta_{k+1}^P = P^{\Sigma_k^P} \subseteq NP^{\Sigma_k^P} = \Sigma_{k+1}^P$.
$\Delta_{k+1}^P = \overline{\Delta}_{k+1}^P \subseteq co\ \Sigma_{k+1}^P = \Pi_{k+1}^P$.
Thus, $\Delta_{k+1}^P \subseteq \Sigma_{k+1}^P \cap \Pi_{k+1}^P$.

iii) The last inclusion can be proven by induction on $k \in \mathbb{N}$. Base case is $\Sigma_0^P = P \subseteq PSPACE$. Now, assume that $\Sigma_k^P \subseteq PSPACE$ is true. Using the above stated fact that $NP^{PSPACE} = PSPACE$, we can see that $\Sigma_{k+1}^P = NP^{\Sigma_k^P} \subseteq PSPACE$. Thus, $\Sigma_k^P \subseteq PSPACE$ for all $k \in \mathbb{N}$. This also means that the whole hierarchy PH is included in $PSPACE$.

Closure properties

The $\Sigma_k^P, \Pi_k^P, \Delta_k^P$ classes are all closed under language union and intersection. We will sketch the proof that Σ_k^P is closed under union.

Suppose

$$A = \mathcal{L}(M_1^C), \quad C \in \Sigma_k^P,$$
$$B = \mathcal{L}(M_2^D), \quad D \in \Sigma_k^P,$$

$$A \cup B = \mathcal{L}(M_3^J), \quad J \in \Sigma_k^P.$$

The idea is to use proof by induction on $k \in \mathbb{N}$. The base case is immediate since $\Sigma_0^P = P$. We then assume that Σ_k^P is closed under union and let A and B be in Σ_{k+1}^P with C, D in Σ_k^P. Then, the "join"

$$J = \{0x \mid x \in C\} \cup \{1y \mid y \in D\}$$

is in Σ_k^P which allows us to prove that $A \cup B \in \Sigma_{k+1}^P$. The closure parallels that of NP languages (i.e., the union of two NP languages is itself a NP language).

Future themes

In the following lectures, we will provide alternative characterizations of Σ_k^P, Π_k^P classes. Using these alternative characterizations we will demonstrate how the polynomial hierarchy could collapse. In particular, we will prove that

$$\begin{aligned} P = NP &\implies PH = P \\ NP = coNP &\implies PH = NP \end{aligned}$$

More generally,

$$\begin{aligned} \Sigma_k^P = \Delta_k^P &\implies PH = \Delta_k^P \\ \Sigma_k^P = \Pi_k^P &\implies PH = \Sigma_k^P \end{aligned}$$

One reason of why we are interested in the polynomial hierarchy is the existence of languages that we do not know how to construct in P or NP, but know how to do in, for example, Σ_2^P.

Alternative characterizations

We will now provide an alternative definition of the class Σ_k^P.

Let $k \geq 1$. The language L is in Σ_k^P if and only if there exist a language $B \in \Pi_{k-1}^P$ and a polynomial function $q(n)$ such that

$$L = \{x \mid \exists y, |y| \leq q(|x|), \text{ and } \langle x, y \rangle \in B\}.$$

This is a generalization of the alternative definition of NP in terms of certificates. In fact,

The language L is in $NP = \Sigma_1^P$ if and only if there exist a language $B \in P = \Pi_0^P$ and a polynomial function $q(n)$ such that

$$L = \{x \mid \exists y, |y| \leq q(|x|), \text{ and } \langle x, y \rangle \in B\}.$$

The theorem above, presented as an alternative characterization of Σ_k^P, can be proven by induction. The "backward" direction is quite trivial; the "forward" direction is presented on pp. 81-82 in Du and Ko [**?**].

Another characterization of Σ_k^P that is sometimes provided as a definition, uses alternative quantifiers and goes as follows.

Let $k \geq 1$. The language L is in Σ_k^P if and only if there exist a language $B \in P$ and a polynomial function $q(n)$ such that

$$\begin{aligned} L = \{x \mid \quad & \exists y_1, |y_1| \leq q(|x|), \\ & \forall y_2, |y_2| \leq q(|x|), \\ & \vdots \\ & Q_k y_k, |y_k| \leq q(|x|), \\ & \text{and } \langle x, y_1, \ldots y_k \rangle \in B\}, \end{aligned}$$

where $Q_k = \begin{cases} \forall, \text{if k is even,} \\ \exists, \text{if k is odd.} \end{cases}$

References

[1] Ding-Zhu Du and Ker-I Ko. *Theory of Computational Complexity*. John Wiley & Sons, Inc., New York, 2000.