1. Let $A \subseteq \Sigma^*$ be $NP$-complete.

   (a) If $B \in P$, $A \cap B = \emptyset$, and $A \cup B \neq \Sigma^*$. Prove that $A \cup B$ is $NP$-complete with respect to $\leq_{polytime}$.

   (b) Is the condition $A \cap B = \emptyset$ necessary in part (a).

2. Consider the graph colouring optimization problem $GCO$ and its natural decision problem $GCD$.

   (a) Show that $GCO \leq_{polytime}^{T} GCD$.

   (b) Given that it is $NP$-hard to decide if a graph is $3$-colourable, show that it is $NP$-hard to obtain a polynomial time $c$-approximation algorithm for any constant $c$.

   Note: It is actually $NP$-hard to obtain a polynomial time $n^{1-\epsilon}$-approximation for any $\epsilon > 0$ but that is a difficult result using the PCP framework. The problem here can be solved using some relatively elementary graph theoretic ideas.

3. Consider the class $BPP^A$, the class of all languages accepted by a polynomial time bounded probabilistic TM with oracle $A$ and with error probability $\leq 1/4$. Suppose $A \in BPP$. Show that $BPP^A = BPP$.

4. More to follow.