CSC 238S **Term Test #1** **February 12/96**

Name ______________________ Student No. ______________________

Tutorial SF1101 ____________ GB120 ____________ GB 305 ____________

Open Book

Answer ALL questions on test paper. Use backs of sheets for scratch work.

Total Marks: 30

1. Consider the following inductive definition of an arithmetic expression:

   BASIS. The following types of atomic operands are arithmetic expressions:

   (i) Variables (ii) Integers (iii) Real numbers

   INDUCTION. If E_1 and E_2 are arithmetic expressions, then (E_1 op E_2) is an arithmetic expression where op is one of $+, -, \times, /$.

   We can inductively define the *depth* of such an arithmetic expression as follows:

   Basis: An atomic operand has depth 0.

   Induction: An expression (E_1 op E_2) has depth $=$ max{depth E_1, depth E_2} $+ 1$.

   Prove:

[5] (i) An expression of depth $d \geq 0$ has at most 2^d atomic operands.

[5] (ii) An expression of depth $d \geq 0$ has at least $d + 1$ atomic operands.

[10] 2. Consider the following recurrence:

$T(0) = 0$

$T(n) = T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + n \quad \text{for } n \geq 1$

where $\lfloor x \rfloor$ is the floor of x.

Prove by induction that $T(n)$ is $O(n)$.

Hint: Use the method of "Solving Recurrences by Guessing a Solution" and note that $c(\lfloor x \rfloor) \leq cx$ for any positive constant c.

3. Consider the following program which, as it turns out, returns n^m whenever $m \in \mathbb{N}$ and $n \in \mathbb{Z}$.

```
POWER(m, n)
    x := m
    y := n
    z := 1
    loop
        % n^m = z · y^x
        exit when x = 0
        if x mod 2 = 1 then z := z · y end if
        x := x div 2
        y := y * y
    end loop
    return z
end POWER
```

[8] (i) Show that $[n^m = z \cdot y^x]$ is a loop invariant.

[2] (ii) Conclude that upon termination of the loop $z = n^m$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}$. (You do not have to prove termination.)