Computer Science CSC-238F — October 4, 1995
Erindale Campus — University of Toronto

Homework Assignment #2
Due: Wednesday, 18 October 1995, 2:10 PM

1. [10 points] Prove that if x and y are nonnegative integers, then the following program eventually halts. (You should describe a quantity which decreases each time through the loop and is bounded below by 0.)

```
loop
    exit when x = 0 and y = 0
    if x > 0 then
        x := x - 1
    else
        x := 100
        y := y - 1
    end if
end loop
```

2. [10 points] Give a DNF formula F with variables p_1, p_2, p_3, p_4, p_5 such that F is true if either less than 2 variables are true or more than 4 variables are true. Give an equivalent CNF formula. (Make sure you explain how you obtained your formulas.)

3. [10 points] Let x and y be three-bit natural numbers. We can represent the value of x by using three propositional variables: x_2 (to denote the value of the most significant bit of x), x_1 (to represent the middle bit of x), and x_0 (to denote the value of the least significant bit of x). Similarly, we can represent the value of y by using three propositial variables y_2, y_1, and y_0, and the sum, $x + y$, can be represented as a four-bit natural number.

For each of the four bits in the sum $x + y$, write a propositional formula (with propositional variables x_0, x_1, x_2, y_0, y_1, and y_2) which expresses the value of that bit. You should write four formulas F_0, F_1, F_2, and F_3 where F_0 expresses the value of the low-order bit of the sum and F_3 expresses the value of the high-order bit. Explain how you obtained your propositional formulas.

EXAMPLE: Consider the truth assignment which gives x the value 6 and y the value 5. That is, the variables x_2, x_1, x_0, y_2, y_1, y_0 get, respectively, the values $1, 1, 0, 1, 0, 1$. Since the sum $6 + 5 = 11$, the values of the formulas F_3, F_2, F_1, F_0 under this truth assignment should be (respectively) $1, 0, 1, 1$.

P.T.O.

4. [10 points]

(a) Let X be a finite set of even size, and let T and T' be two subsets of X, both of which have even size. Also, let $\overline{T}$ and $\overline{T}'$ be the complements of T and T' with respect to X.

Prove that either all of $T \cap T'$, $\overline{T} \cap T'$, $T \cap \overline{T}'$, $\overline{T} \cap \overline{T}'$ have even size or all of them have odd size.

(b) The binary Boolean connectives $\leftrightarrow$ and $\oplus$ are defined by the following truth-table:

P	Q	$P \oplus Q$	$P \leftrightarrow Q$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

Let $k \geq 2$ and consider formulas using no connectives other than $\leftrightarrow$ and $\oplus$, and using only variables from amongst $x_1, \ldots, x_k$ (not all variables need occur).

Show that for any such formula F, the number of truth assignments to $\{x_1, \ldots, x_k\}$ which satisfy F is even. (Hint: Use induction and part (a)).

(c) Prove that $\{\leftrightarrow, \oplus\}$ is not a complete set of connectives.

5. [EXTRA CREDIT PROBLEM] **WARNING:** This question is *very* difficult. It is put here to show you that it may be hard to prove termination of even a very simple program.

Prove that if x is a positive integer, then the following program eventually halts.

```
loop
    exit when x = 1
    if x is even then
        x := x/2
    else
        x := 3 × x + 1
    end if
end loop
```