University of Toronto
Faculty of Arts and Science
April/May Examinations 1996
CSC 238S

Duration - 3 hours
Aids Allowed: Open Book
There are 7 questions worth a total of 85 marks.
Answer all questions on the question paper, using backs of pages for scratch work.
Check that your exam book has 9 pages (including this cover page).

PLEASE COMPLETE THIS SECTION:

Name ______________________________
(Please underline your family name.)

Student Number ______________________________

FOR USE IN MARKING:

1. ____________/20

2. ____________/10

3. ____________/10

4. ____________/10

5. ____________/20

6. ____________/ 5

7. ____________/10

Total:__________/ 85

1. Consider the following recursive subprogram:

```
function testgrowth(n :int)
    if n ≤ 1 then result 1
    else result := 2* testgrowth(n − 2)
end testgrowth
```

[5] (a) Write a recurrence that characterizes the value of *testgrowth*(n) for $n \geq 0$.

[5] (b) Solve the recurrence and prove an asymptotically optimal upper bound for *testgrowth*(n) for $n \geq 0$.

[5] (c) Write a recurrence that characterizes $T(n)$, for $n \geq 0$. Here $T(n)$ is the time complexity (= number of instructions executed) for computing *testgrowth*(n).

[5] (d) Solve the recurrence and prove an asymptotically optimal upper bound for $T(n)$ for $n \geq 0$.

[10] 2. Let

$$\begin{array}{ll} T(n) = 0 & \text{for } n = 0, 1, 2, 3 \\ T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor 3n/4 \rfloor) + n & \text{for } n \geq 4 \end{array}$$

Prove that $T(n) = O(n \log n)$.

Hint: Solve the recurrence by guessing a solution and note that $c\lfloor x \rfloor \log(\lfloor x \rfloor) \leq cx \log x$ for any positive constant c. If it helps, you may use -.4 to approximate $\log(3/4)$.

[10] 3. Prove that $\{\oplus, \rightarrow\}$ is a functionally complete set of propositional connectives.

Hint: $p \rightarrow (p \oplus p) \equiv ?$

[10] 4. Consider the following interpretation. Assume that the Domain is the set of non negative integers, and that the following symbols exist and are interpreted as follows.

predicates:	EQ(X, Y)	signifies	"$X = Y$"
	GE(X, Y)	signifies	"$X \geq Y$"
	MULT(X, Y, Z)	signifies	"$X \times Y = Z$"
constants:	0,1,2		

Write an expression in predicate logic to represent each of the following:

(a) Y divides X exactly (i.e. write a formula with two free variables X and Y).

(b) X is a power of 2 (i.e. write a formula with one free variable X). You may use DIVIDE(Y, X) to denote "Y divides X exactly".

Note: You must use the above predicates and constants (and DIVIDES), but you cannot write 2^K.

5. Assume (as in Term Test 2) we have a set of individuals as a Domain, and that the following symbols exist and are interpreted as follows.

predicates:	PARENT(X, Y)	signifies	"X is a biological parent of Y"
	M(X)	signifies	"X is a male"
	F(X)	signifies	"X is a female"

Consider the following "axioms":

$$
\begin{array}{ll}
F_1 & \forall X\,((\mathrm{M}\ (X) \vee \mathrm{F}\ (X)) \wedge (\mathrm{M}(X) \rightarrow \neg \mathrm{F}(X)) \\
F_2 & \forall Y \exists X_1 \exists X_2\,(\mathrm{PARENT}\ (X_1, Y) \wedge \mathrm{PARENT}\ (X_2, Y) \wedge \mathrm{M}\ (X_1) \wedge \mathrm{F}\ (X_2)) \\
F_3 & \forall X_1 \forall X_2 \forall Y\,((\mathrm{PARENT}\ (X_1, Y) \wedge \mathrm{PARENT}\ (X_2, Y) \wedge X_1 \neq X_2) \\
 & \qquad\qquad \rightarrow ((\mathrm{M}\ (X_1) \rightarrow \mathrm{F}\ (X_2)) \wedge (\mathrm{F}\ (X_1) \rightarrow \mathrm{M}\ (X_2)))) \\
F_4 & \forall X \forall Y (\mathrm{PARENT}(X, Y) \rightarrow \neg\ \mathrm{PARENT}(Y, X)))
\end{array}
$$

(Note: F_1, F_2, F_3 are the same axioms used in Term Test 2; F_4 is a new axiom.)

We have (as in Term Test 2) the following definitions:

GRANDPARENT (X, Y) :

$$\exists Z\,(\mathrm{PARENT}\ (Z, Y) \wedge \mathrm{PARENT}\ (X, Z))$$

COUSIN (X, Y):

$$\exists Z\,(\mathrm{GRANDPARENT}\ (Z, X) \wedge \mathrm{GRANDPARENT}\ (Z, Y))$$

Which of the following logically follows from the axioms F_1, F_2, F_3, F_4; i.e. in any interpretation making F_1, F_2, F_3 and F_4 true, the statement is true. Explain your answer briefly. Note that an interpretation need not correspond to "reality".

[5] i) An individual has at least 3 grandparents

[5] ii) The COUSIN relation is transitive; i.e.

$$\forall X \forall Y \forall Z((\text{COUSIN }(X, Y) \wedge \text{ COUSIN }(Y, Z)) \rightarrow \text{COUSIN }(X, Z))$$

[5] iii) No one is their own parent; i.e.

$$\forall X \neg \text{PARENT }(X, X)$$

[5] iv) The domain must be infinite.

[5] 6. Give a 4 state deterministic FSA for the language described by the following regular expression:

$$00^* \mid 00^*(\ 11\ )^*.$$

[10] 7. Let L_1 and L_2 be languages. Define

$$L_1 \oplus L_2 = \{w \mid (w \in L_1 \text{ and } w \notin L_2) \text{ or } (w \notin L_1 \text{ and } w \in L_2)\ \}$$

Briefly prove the following using known facts (including facts proven in problem sets) about regular languages.

If L_1 and L_2 are regular, then $L_1 \oplus L_2$ is regular.