1. We assume that $T(b^k) \geq c \cdot g(b^k)$ for k sufficiently large. Let n be sufficiently large and let k be such that $b^{k-1} < n \leq b^k$.

$$\begin{aligned} T(n) &\geq T(b^{k-1}) \text{ since } T \text{is non decreasing} \\ &\geq c \cdot g(b^{k-1}) \\ &= c \cdot g(\tfrac{1}{b} \cdot b^k) \\ &\geq c \cdot c' \cdot g(b^k) \\ &\geq (c \cdot c') \cdot g(n) \text{ since } g \text{ is non decreasing} \end{aligned}$$

2. a) $n! = \Pi_{i=1}^{n} i > \Pi_{i=n/2}^{n} i \geq (\frac{n}{2})^{n/2}$. Therefore, $\log(n!) \geq \frac{n}{2} \log \frac{n}{2} = \frac{n}{2}(\log n - 1)$ and $\lim \frac{\log(n!)}{n \log n} \geq \frac{1}{2}$. Hence $n!$ is $\Omega(n \log n)$.

b)
$$\begin{aligned} \lim \tfrac{3^n}{n!} &\leq \lim \tfrac{3^n}{(\frac{n}{2})^{x/2}} \\ &= \lim \tfrac{9^{n/2}}{(\frac{n}{2})^{n/2}} \\ &= \lim (\tfrac{18}{n})^{n/2} = 0 \end{aligned}$$

3. We'll guess $T(n) \leq c \cdot n$ for all $n \geq n_0$ for some appropriate constants c, d, n_0. Note that for $n_0 \geq 1$, we can always choose c sufficiently large so as to establish the basis for an induction.

Suppose $T(k) \leq c \cdot k$ for all k, satisfying $n_0 \leq k < n$

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + n \\ &\leq c \cdot \lfloor n/2 \rfloor + c \cdot \lfloor n/4 \rfloor + n \\ &\leq \tfrac{3}{4} \cdot c \cdot n + n \\ &= (\tfrac{3}{4}c + 1) \cdot n \end{aligned}$$

We want $\frac{3}{4}c + 1 \leq c$ which holds when $1 \leq \frac{c}{4}$. So we can prove by induction that $T(n) \leq 4 \cdot n$ for all $n \geq 1$. (The base case is easily established as $1 = T(1) \leq 4 \cdot 1 = 4$.)

4. a) $A \wedge B$
b) $A \wedge (B \vee C)$
c) $B \rightarrow (A \wedge C)$
d) $A \rightarrow (\neg B \vee C)$
e) $A \wedge (C \rightarrow (\neg B \vee C)$

5. a) $x \oplus x$
b) $x \rightarrow (x \oplus x)$
c) Need to show how to realize the connective $\{\wedge, \vee, \neg\}$. (In fact, it is sufficient to realize $\{\vee, \neg\}$ using De Morgue's Laws). We already have shown how to realize $\neg$, so all that remains is $\vee$.

$$\begin{aligned} y \vee z &\equiv \neg y \rightarrow z \\ &\equiv (y \rightarrow (y \oplus y)) \rightarrow z \end{aligned}$$

6. For definiteness, let's say that the length of Q is the number of connectives occurring in Q.

Basis (no connectives): Q is either x or the constant symbol TRUE, so the basis holds trivially.

Induction step:

Case i) Q is $Q_1 \vee Q_2$

If $Q_1 \equiv$ true or $Q_2 \equiv$ true then

$Q \equiv$ true

If $Q_1 \equiv x$ and $Q_2 \equiv x$ then

$Q \equiv x \vee x \equiv x$.

Case ii) Q is $Q_1 \rightarrow Q_2$

Q_1	Q_2	$Q_1 \rightarrow Q_2$
TRUE	TRUE	TRUE
TRUE	x	x
x	TRUE	x
x	x	TRUE