Computer Science 238 November 11, 1998
St. George Campus University of Toronto

Homework Assignment #4
Due: Wednesday, November 25, 1998

On the cover page of your assignment, you must write ***and sign*** *the following statement: "I have read and understood the policy on collaboration in homework stated in the Course Information handout." Without such a signed statement your homework will not be marked.*

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1. (10 points) Recall that there are $2^{(2^n)}$ Boolean functions of n variables. For each of the sixteen Boolean functions of two variables x_1 and x_2, write a propositional formula that represents that function. Your formulas may use any of the Boolean connectives discussed in the notes.

2. (10 points)

(a) (1 point) Let P be any formula and Q be any unsatisfiable formula. Prove that $P \to Q$ is logically equivalent to $\neg P$.

(b) (9 points) Prove that $\{\oplus, \to\}$ is a complete set of Boolean connectives. (Hint: Use part (a) to express $\neg$ using $\oplus$ and $\to$.)

3. (10 points) Consider a first-order language with an unlimited supply of variables, two binary predicate symbols P and $=$, and no constant symbols. $P(x, y)$ is intended to be interpreted as "x is a parent of y", and $=(x, y)$ is the equality predicate.

(a) Write formulas in this language to express the following three predicates:

(i) $Sib(x, y)$: x is a (full) sibling of y — i.e., x has exactly the same parents as y.

(ii) $HalfSib(x, y)$: x is a half sibling of y — i.e., x has *exactly one* parent in common with y.

(iii) $Cousin(x, y)$: x is a cousin of y — i.e., x has (at least) one grandparent in common with, but is not a full sibling of, y. (Note: This is not the "usual" definition of cousin in which half siblings are not considered cousins.)

(b) Write sentences (i.e., first-order formulas that have no free variables) to express each of the statements below. Your sentences may use the formulas Sib, $HalfSib$ and $Cousin$ (with appropriate variables) that you constructed in part (a). If you find that some sentence is ambiguous in English, you should point out that ambiguity and restate the sentence (in English) so as to resolve all ambiguities.

(i) Everyone has exactly two distinct parents.

(ii) No person has a grandparent who is that person's cousin.

(iii) There is someone all of whose cousins are his/her half-siblings and who actually has (at least) one cousin.

(iv) Everyone who has a cousin has at least one grandparent who has (at least) two distinct children.

(v) The half-siblings of any person are also half-siblings of that person's full siblings.

4. (10 points) Transform the following first-order formula into an equivalent PNF formula in which the quantifier-free part uses only the connectives $\wedge$, $\vee$ and $\neg$, and every occurrence of $\neg$ is applied to an *atomic* formula. Demonstrate the steps through which you obtain your final formula.

$$(\forall x\, \neg \exists y\, A(x,y,z) \rightarrow \neg \forall x\, \exists y\, B(x,y,z)) \rightarrow \forall x\, (C(x) \wedge \exists z\, D(x,z))$$

5. (EXTRA CREDIT — worth 10 points) Let F be the formula $(F_1 \wedge F_2 \wedge F_3)$ where:

$$\begin{array}{ll} F_1 \text{ is:} & \forall x\, \forall y \forall z\, ((L(x,y) \wedge L(y,z)) \rightarrow L(x,z)) \\ F_2 \text{ is:} & \forall x\, \exists y\, L(x,y) \\ F_3 \text{ is:} & \forall x\, \neg L(x,x) \end{array}$$

(a) Describe a structure that satisfies F. (HINT: Consider "less than".)

(b) Prove that every structure that satisfies F must have an *infinite* domain.

(c) Prove that there is a structure with a finite domain that satisfies $(F_1 \wedge F_2)$.