Computer Science CSC 238F | November 25 1998
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Solutions to Homework Assignment #4

1. There are many ways to answer this question correctly. The following table gives one possible formula for each boolean function $f(x_1, x_2)$.

$f(0,0)$	$f(0,1)$	$f(1,0)$	$f(1,1)$	formula for f
0	0	0	0	$x_1 \wedge \neg x_1$
0	0	0	1	$x_1 \wedge x_2$
0	0	1	0	$x_1 \wedge \neg x_2$
0	0	1	1	x_1
0	1	0	0	$\neg x_1 \wedge x_2$
0	1	0	1	x_2
0	1	1	0	$x_1 \oplus x_2$
0	1	1	1	$x_1 \vee x_2$
1	0	0	0	$\neg(x_1 \vee x_2)$
1	0	0	1	$\neg(x_1 \oplus x_2)$
1	0	1	0	$\neg x_2$
1	0	1	1	$\neg(\neg x_1 \wedge x_2)$
1	1	0	0	$\neg x_1$
1	1	0	1	$\neg(x_1 \wedge \neg x_2)$
1	1	1	0	$\neg(x_1 \wedge x_2)$
1	1	1	1	$\neg(x_1 \wedge \neg x_1)$

2. (a) Since Q is unsatisfiable, it is logically equivalent to $R \wedge \neg R$. So,

$$\begin{aligned} P \rightarrow Q \quad & \text{LEQV} \quad \neg P \vee Q && (\rightarrow \text{ law}) \\ & \text{LEQV} \quad \neg P \vee (R \wedge \neg R) && (\text{Theorem 3.9}) \\ & \text{LEQV} \quad \neg P && (\text{identity law}) \end{aligned}$$

(b) We shall show that $\neg$ and $\vee$ can be expressed using $\oplus$ and $\rightarrow$. Let P and R be any propositional formulae. First, notice that $R \oplus R$ is false regardless of the truth value of R. By part (a), $\neg P$ LEQV $P \rightarrow (R \oplus R)$. We also have

$$\begin{aligned} P \vee R \quad & \text{LEQV} \quad \neg\neg P \vee R && (\text{double negation}) \\ & \text{LEQV} \quad \neg P \rightarrow R && (\rightarrow \text{ rule}) \\ & \text{LEQV} \quad (P \rightarrow (R \oplus R)) \rightarrow R && (\text{part (a)}) \end{aligned}$$

By Theorem 3.22, every boolean function can be represented using the connectives $\neg$ and $\vee$. We showed here that each of these two connectives can be expressed using $\rightarrow$ and $\oplus$. Thus, every boolean function can be represented using $\rightarrow$ and $\oplus$. (One could prove this formally by mathematical induction, as in the proof of Theorem 3.21 in the text.) So, $\{\oplus, \rightarrow\}$ is a complete set of connectives.

3. (a) (i) $Sib(x,y) : \forall z(P(z,x) \leftrightarrow P(z,y))$.

(ii) $HalfSib(x,y) : \exists z \forall w[(P(w,x) \wedge P(w,y)) \leftrightarrow = (z,w)]$.

(iii) Let's introduce another formula, $Grandparent(x,y) : \exists z[P(x,z) \wedge P(z,y)]$, which says "$x$ is a grandparent of y". Then we can express the cousin predicate as $Cousin(x,y) : \exists u[Grandparent(u,x) \wedge Grandparent(u,y)] \wedge \neg Sib(x,y)$.

(b) (i) $\forall x \exists y \exists z[\neg = (y,z) \wedge \forall w[P(w,x) \leftrightarrow (= (w,y) \vee = (w,z))]$.

(ii) $\neg \exists x \exists y[Grandparent(y,x) \wedge Cousin(y,x)]$.

(iii) $\exists x[\forall y(Cousin(y,x) \rightarrow HalfSib(y,x)) \wedge \exists z Cousin(z,x)]$.

(iv) $\forall x[\exists y Cousin(y,x) \rightarrow \exists z(Grandparent(z,x) \wedge \exists u \exists v(P(z,u) \wedge P(z,v) \wedge \neg = (u,v)))]$.

(v) $\forall x \forall y[HalfSib(y,x) \rightarrow \forall z(Sib(z,x) \rightarrow HalfSib(y,z))]$.

4. In the first step, we shall rename variables so that each quantifier has a distinct variable. The Roman numerals refer to rules on pages 133-134 of the course notes.

	$(\forall x \neg \exists y A(x,y,z) \rightarrow \neg \forall x \exists y B(x,y,z)) \rightarrow \forall x(C(x) \wedge \exists z D(x,z))$	
LEQV	$(\forall x \neg \exists y A(x,y,z) \rightarrow \neg \forall t \exists u B(t,u,z)) \rightarrow \forall v(C(v) \wedge \exists w D(v,w))$	**(III)**
LEQV	$\neg(\neg \forall x \neg \exists y A(x,y,z) \vee \neg \forall t \exists u B(t,u,z)) \vee \forall v(C(v) \wedge \exists w D(v,w))$	($\rightarrow$ rule)
LEQV	$(\neg\neg \forall x \neg \exists y A(x,y,z) \wedge \neg\neg \forall t \exists u B(t,u,z)) \vee \forall v(C(v) \wedge \exists w D(v,w))$	(DeMorgan)
LEQV	$(\forall x \neg \exists y A(x,y,z) \wedge \forall t \exists u B(t,u,z)) \vee \forall v(C(v) \wedge \exists w D(v,w))$	(double negation)
LEQV	$(\forall x \forall y \neg A(x,y,z) \wedge \forall t \exists u B(t,u,z)) \vee \forall v(C(v) \wedge \exists w D(v,w))$	**(Ib)**
LEQV	$\forall x \forall y \forall t \exists u(\neg A(x,y,z) \wedge B(t,u,z)) \vee \forall v \exists w(C(v) \wedge D(v,w))$	(**IIa**, applied five times)
LEQV	$\forall x \forall y \forall t \exists u \forall v \exists w[(\neg A(x,y,z) \wedge B(t,u,z)) \vee (C(v) \wedge D(v,w))]$	(**IIb**, applied six times)

5. **(a)** Consider the structure whose domain is $\mathbb{N}$ where $L(x,y)$ is interpreted to mean "$x < y$". It's easy to check that this structure satisfies F.

(b) Let $\mathcal{S}$ be any structure that satisfies F. We shall inductively define an infinite sequence of elements of the domain of $\mathcal{S}$ and then show that all elements in this sequence are distinct.

Base case: Let e_0 be an arbitrary element of the domain. (We can choose such an element, since the domain of any structure is non-empty.)

Inductive step: Suppose we have defined the element e_i. We wish to define e_{i+1}. The structure $\mathcal{S}$ satisfies F_2. Thus there is some element q in the domain such that $(e_i, q) \in L^{\mathcal{S}}$. Let e_{i+1} be the element q.

Let $P(n)$ be the statement "$(e_i, e_{i+n}) \in L^{\mathcal{S}}$ (for all $i \in \mathbb{N}$)." We shall show that $P(n)$ is true for all $n > 0$ by mathematical induction on n.

Base case: When $n = 1$, we know that $(e_i, e_{i+1}) \in L^{\mathcal{S}}$ by the definition of e_{i+1}.

Inductive step: Let $n > 1$. Suppose $P(n-1)$ is true. Let $i \in \mathbb{N}$. By the definition of e_{i+1}, $(e_i, e_{i+1}) \in L^{\mathcal{S}}$. By the induction hypothesis, $(e_{i+1}, e_{i+n}) \in L^{\mathcal{S}}$. Consider a valuation σ that has $\sigma(x) = e_i, \sigma(y) = e_{i+1}$ and $\sigma(z) = e_{i+n}$. Since the structure $\mathcal{S}$ satisfies F_1, the interpretation $(\mathcal{S}, \sigma)$ satisfies $(L(x,y) \wedge L(y,z)) \rightarrow L(x,z)$. But $L(x,y)$ and $L(y,z)$ are true under this interpretation, so $L(x,z)$ must also be true. Thus, we have $(e_i, e_{i+n}) \in L^{\mathcal{S}}$.

This completes the proof of $P(n)$ for all $n > 0$.

It follows that $e_i \neq e_j$ for all $i, j \in \mathbb{N}$ with $j > i$, for if e_i were equal to e_j, we would have $(e_i, e_j) = (e_i, e_i) \in L^{\mathcal{S}}$ (because $P(j-i)$ is true) and this contradicts the fact that $\mathcal{S}$ satisfies F_3.

Thus, we have an infinite sequence of elements in the domain of $\mathcal{S}$, no two of which are equal, so the domain must be infinite.

(c) Consider the structure $\mathcal{S}$ with domain $\{1\}$ and $L^{\mathcal{S}} = \{(1,1)\}$. It's easy to check that this structure satisfies F.