CSC 238F 1998 Homework Assignment 3, Due Nov. 11

On the cover page of your assignment, you must write **and sign** *the following statement: "I have read and understood the policy on collaboration in homework stated in the Course Information handout." Indicate the name of your TA and/or your tutorial room.*

1. *Prove that the following recursive procedure $D(x, y, q, r)$ is correct with respect to the following Precondition/Postcondition pair.*

   Precondition: $x, y \in \mathbb{N}$, $y > 0$.

   Postcondition: $D(x, y, q, r)$ *halts, and* q *has the value* $(x \text{ div } y)$.

   $D(x, y, q, r)$
       if $(x = 0)$ **then**
           $q := 0;\quad r := 0$
       else
           $D(x \text{ div } 2, y, q', r')$
           if $(2 * r' + x \bmod 2 < y)$ **then**
               $q := 2 * q';\quad r := 2 * r' + x \bmod 2$
           else
               $q := 2 * q' + 1;\quad r := 2 * r' + x \bmod 2 - y$
           end if
       end if
   end D

   (HINT: Your induction hypothesis should involve the value assigned to r as well as q.)

2. *Consider the following recurrence:*

$$T(n) = \begin{cases} 10 & \text{if } n = 0, \\ 100\,T(\lfloor n/5 \rfloor) + n^2 3^n & \text{if } n > 0. \end{cases}$$

   Prove that $T(n)$ is $\Theta(n^2 3^n)$; you should not *use the General Theorem about divide-and-conquer recurrences, but rather prove this theorem directly. (Note: it is not sufficient to just prove this for powers of 5, since for general n, this would merely yield the worse bound $\mathcal{O}(n^2 3^{5n})$.)*

3. *Let P, Q and R be propositional formulas so that P is a* **tautology**, *Q is* **unsatisfiable** *and R is a* **satisfiable formula that is not a tautology**.
   NOTE: A formula is said to be satisfiable if there is some truth assignment that makes it true; a formula is said to be unsatisfiable if there is no truth assignment that makes it true.

   For each of the statements (a)–(d) below, indicate whether it is true or false, and justify your answer. **(No credit will be given without correct justification!)**

(a) *P is logically equivalent to $\neg Q$.*

(b) *$P \rightarrow Q$ is satisfiable.*

(c) *$Q \rightarrow R$ is a tautology.*

(d) *$R \rightarrow Q$ is satisfiable.*

4. *Consider truth assignments involving only the propositional variables $x_0, x_1, y_0, y_1, y_2, y_3$. Every such truth assignmnent gives a 1 (representing true) or 0 (representing false) value to each variable; we can therefore think of a truth assignment τ as determining a two bit integer value x_τ and a four bit integer value y_τ, depending on the two bit values given x_0 and x_1, and the four bit values given y_0, y_1, y_2, y_3. More formally we can define the integers $x_\tau = \tau(x_0) + 2 * \tau(x_1)$ and $y_\tau = \tau(y_0) + 2 * \tau(y_1) + 4 * \tau(y_2) + 8 * \tau(y_3)$.*

   For this question, you are to write a formula F which is true in exactly those truth assignments τ for which $y_\tau = (x_\tau)^2$. Your formula should either be in DNF or CNF.