Solutions to Homework Assignment #3

1. As suggested in the assignment, we shall actually use a stronger postcondition, which says that the quotient and remainder obtained when dividing x by y are stored in q and r, respectively.

Let $P(x)$ be the statement "If $D(x, y, q, r)$ is called with $y > 0$, then it terminates with $x = qy+r$ and $0 \leq r < y$." We shall prove that $P(x)$ is true for all natural numbers x by strong induction on x. (One could also set up the proof to use ordinary induction on the length of the binary representation of x.)

Basis: When $x = 0$ (and $y > 0$), the call $D(x, y, q, r)$ terminates immediately after setting q and r to 0, so we have $x = qy + r$ and $0 \leq r < y$. Thus $P(0)$ is true.

Inductive Step: Let $x > 0$. Suppose $P(i)$ is true for $0 \leq i < x$. We shall prove that $P(x)$ is true.

Suppose $D(x, y, q, r)$ is called with $y > 0$. We have $0 \leq x \text{ div } 2 < x$, and we know that the inputs for the recursive call $D(x \text{ div } 2, y, q', r')$ satisfy the precondition. Hence, by the induction hypothesis, the recursive call terminates with $x \text{ div } 2 = q'y + r'$ and $0 \leq r' < y$. Clearly, the main call $D(x, y, q, r)$ terminates, since the rest of the code is executed in constant time.

We must now show that the postconditions are satisfied. Let k denote $x \text{ div } 2$ and j denote $x \bmod 2$. Then we have $x = 2k + j$ and $0 \leq j < 2$.

Case I $(2r' + j < y)$:
Then, $q = 2q'$ and $r = 2r' + j$. So,

$$\begin{aligned} x &= 2k + j && \text{(by definition of } k \text{ and } j) \\ &= 2(q'y + r') + j && \text{(proved above)} \\ &= (2q')y + (2r' + j) \\ &= qy + r \end{aligned}$$

and $r = 2r' + j$ satisfies $0 \leq r < y$.

Case II $(2r' + j \geq y)$:
Then, $q = 2q' + 1$ and $r = 2r' + j - y$. So,

$$\begin{aligned} x &= 2k + j && \text{(by definition of } k \text{ and } j) \\ &= 2(q'y + r') + j && \text{(proved above)} \\ &= (2q' + 1)y + (2r' + j - y) \\ &= qy + r. \end{aligned}$$

The assumption for Case II implies that $r = 2r' + j - y$ is at least 0. Furthermore, since $r' \leq y - 1$ and $j \leq 1$, we have $r \leq 2(y-1) + j - y = y + j - 2 < y$.

This completes the proof of $P(x)$ for all natural numbers x. In particular, the programme is correct with respect to the given precondition/postcondition pair.

2. It's easy to see that $T(n) \geq 0$ for all $n \geq 0$. (Formally, one could prove this using strong induction on n.) It follows from the recurrence that $T(n) \geq 0 + n^2 3^n$ for all $n > 0$. Hence, $T(n) \in \Omega(n^2 3^n)$.

Now we shall prove an upper bound on $T(n)$. Let $P(n)$ be the statement $T(n) \leq cn^2 3^n$, where we choose the constant c to be 1000 in order to make the following proof work out nicely.

We shall prove that $P(n)$ is true for all $n > 0$ by strong induction on n.

Basis: When $1 \leq n \leq 4$, $T(n) = 100T(0) + n^2 3^n = (\frac{1000}{n^2 3^n} + 1)n^2 3^n \leq 1000n^2 3^n$.

Inductive Step: Let $n \geq 5$. Assume that $P(k)$ is true for $1 \leq k < n$. We shall prove that $P(n)$ is true.

First, notice that since $n \geq 5$, $1 \leq \lfloor n/5 \rfloor \leq n/5 < n$. So, by the inductive hypothesis, $T(\lfloor n/5 \rfloor) \leq cn^2 3^n$. Thus,

$$\begin{aligned}
T(n) &= 100T(\lfloor n/5 \rfloor) + n^2 3^n \qquad \text{(since } n \geq 5 > 0\text{)} \\
&\leq 100c\lfloor n/5 \rfloor^2 3^{\lfloor n/5 \rfloor} + n^2 3^n \\
&\leq 100c(n/5)^2 3^{n/5} + n^2 3^n \qquad \text{(since } \lfloor n/5 \rfloor \leq n/5\text{)} \\
&= 4cn^2 3^{n/5} + n^2 3^n \\
&= (4c/3^{4n/5} + 1)n^2 3^n \\
&\leq (4c/81 + 1)n^2 3^n \qquad \text{(for } n \geq 5,\ 3^{4n/5} \geq 3^{4 \cdot 5/5} = 81\text{)} \\
&\leq cn^2 3^n \qquad \text{(since } 4000/81 + 1 \leq 1000\text{)}.
\end{aligned}$$

This completes the proof that $T(n) \leq 1000n^2 3^n$ for all $n > 0$. Hence, $T(n) \in O(n^2 3^n)$. Since $T(n)$ is also in $\Omega(n^2 3^n)$, we have $T(n) \in \Theta(n^2 3^n)$.

3. **(a)** True. Under all truth assignments to the propositional variables, the formula P is true, the formula Q is false, and hence the formula $\neg Q$ is true. Thus, P and $\neg Q$ get the same truth value under all truth assignments.

(b) False. Under all truth assignments, P is true and Q is false, so $P \rightarrow Q$ is false. Thus, there is no truth assignment that satisfies $P \rightarrow Q$.

(c) True. Under all truth assignments, Q is false, so $Q \rightarrow R$ is true (regardless of the value assigned to R).

(d) True. There is some truth assignment that makes R false (since R is not a tautology), and $R \rightarrow Q$ is satisfied by that assignment.

4. For any truth assignment τ, $x_\tau \in \{0, 1, 2, 3\}$. For each of the possible values of x_τ, there is exactly one value of y_τ that satisfies $y_\tau = x_\tau^2$. Thus, the formula F should be satisfied only by the following four truth assignments:

x_τ	y_τ	x_1	x_0	y_3	y_2	y_1	y_0
0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	1
2	4	1	0	0	1	0	0
3	9	1	1	1	0	0	1

Thus, a formula F that works is

$(\neg x_1 \wedge \neg x_0 \wedge \neg y_3 \wedge \neg y_2 \wedge \neg y_1 \wedge \neg y_0) \vee$
$(\neg x_1 \wedge x_0 \wedge \neg y_3 \wedge \neg y_2 \wedge \neg y_1 \wedge y_0) \vee$
$(x_1 \wedge \neg x_0 \wedge \neg y_3 \wedge y_2 \wedge \neg y_1 \wedge \neg y_0) \vee$
$(x_1 \wedge x_0 \wedge y_3 \wedge \neg y_2 \wedge \neg y_1 \wedge y_0)$.

This formula is in disjunctive normal form.