CSC 238F 1998, Assignment 1
Due: Oct. 7, 10:10 A.M.

On the cover page of your assignment, you must write **and sign** *the following statement: "I have read and understood the policy on collaboration in homework stated in the Course Information handout." Without such a signed statement your homework will not be marked.*

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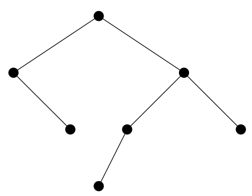
1. Show that $(4^n - 1) \bmod 3 = 0$ for all $n \in \mathbb{N}$. [10]

2. We define the *height* of a (rooted) tree to be the number of edges in a longest path from the root to any leaf. If we define $f(h)$ to be the number of nodes in the "perfectly complete" binary tree (i.e. a binary tree in which there is a maximum number of nodes at each level of the tree) of height h, it is easy to see that f satisfies the following properties: [10]
   $f(0) = 1$ and
   $f(h+1) = 2f(h) + 1$ for every $h \geq 0$.

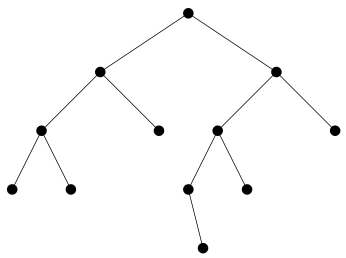
   Use induction to prove that $f(h) \leq 2^{h+1}$ for all $h \geq 0$. (HINT: It may be easier to prove the stronger statement that $f(h) \leq 2^{h+1} - 1$.)

3. An important type of rooted tree for many data structures is a Nearly Balanced Binary Tree (or NBBT). A NBBT is defined to be a binary tree where every node x satisfies the following property:
   If x has two subtrees as children, say T_1 and T_2, then the height of T_1 differs from the height of T_2 by at most one;
   if x has only one subtree T_1, then that subtree must consist of only a single node.

   Examples: The following tree is a NBBT.



   The following tree in *not* a NBBT.



We define the height of such a tree as in the previous quetsion. Define $g(h)$ to be the smallest possible number of nodes in a NBBT of height h.

(a) Prove that [5]
$g(0) = 1$, $g(1) = 2$, and $g(h) = g(h-1) + g(h-2) + 1$ for every $h \geq 2$.
You may assume (without proving it) that g is strictly monotonic, that is, if $h < h'$, then $g(h) < g(h')$.
NOTE: This proof should not involve induction, but only reasoning based on the definition of NBBTs.

(b) Use Part *a* (and complete induction) to prove that $g(h) \geq 1.6^h$ for every $h \geq 0$. [10]
(This result implies that the height of a NBBT is at most logarithmic in its size, an important fact for algorithms operating on NBBTs.)

(c) **Extra Credit** [5]
It turns out that 1.6 is not the largest real number for which Part *b* is true. What is the largest real number c such that for every $h \geq 0$, $g(h) \geq c^h$. Justify your answer.

4. Imagine we live in a country where the only denominations of stamps are 5 and 8, and so we are interested in what values of postage can be obtained by combining (zero or more) stamps of value 5 with (zero or more) stamps of value 8.

   For this question, you should produce a particular positive integer u and prove the following:

   (a) For every integer $n \geq u$, it is possible to form postage n. [5]

   (b) It is *not* possible to form postage $u - 1$. [5]

5. Prove that the following program halts for every input $n \in \mathbb{N}$. [10]

```
y := 3 * n     % Precondition: n is a natural number
loop
    % You should prove a Loop Invariant
    exit when y = 0
    if y mod 4 = 0 then
        y := y-6
    else
        y := y-3
    end if
end loop
```