Computer Science 238 — Fall 1998
University of Toronto

Answers to Homework Assignment #1

ANSWER TO QUESTION 1. Notice that $x \bmod 3 = 0$ is the same thing as "x is a multiple of 3". Let $P(n)$ be the predicate "$4^n - 1$ is a multiple of 3". We'll prove that $P(n)$ holds for all n, by induction.

BASIS: $4^0 - 1 = 1 - 1 = 0$, and 0 is a multiple of 3, so $P(0)$ holds.

INDUCTION STEP: Let i be an arbitrary natural number. Suppose that $P(i)$ holds; i.e., $4^i - 1$ is a multiple of 3. We must show that $P(i+1)$ holds, i.e. $4^{i+1} - 1$ is a multiple of 3. We have:

$$\begin{aligned}
4^{i+1} - 1 &= 4 \cdot 4^i - 1 \\
&= (3 \cdot 4^i + 4^i) - 1 \\
&= 3 \cdot 4^i + (4^i - 1) \\
&= 3 \cdot 4^i + 3k && \text{[for some } k \in \mathbb{N}\text{, since by i.h. } 4^i - 1 \text{ is a multiple of 3]} \\
&= 3 \cdot (4^i + k)
\end{aligned}$$

So, $4^{i+1} - 1$ is a multiple of 3, as wanted.

ANSWER TO QUESTION 2. Let $P(h)$ be the predicate "$f(h) \leq 2^{h+1} - 1$". We will show that $P(h)$ holds for all $h \in \mathbb{N}$, by induction.

BASIS: $f(0) = 1$, by definition of f. $2^{0+1} - 1 = 2 - 1 = 1$. So, $f(0) \leq 2^{0+1} - 1$, as wanted.

INDUCTION STEP: Let i be an arbitrary natural number. Suppose that $P(i)$ holds; i.e., $f(i) \leq 2^{i+1} - 1$. We must show that $P(i+1)$ holds, i.e. $f(i+1) \leq 2^{i+2} - 1$. We have:

$$\begin{aligned}
f(i+1) &= 2 \cdot f(i) + 1 && \text{[by definition of } f\text{, since } i+1 > 0] \\
&\leq 2 \cdot (2^{i+1} - 1) + 1 && \text{[by induction hypothesis]} \\
&= 2^{i+2} - 2 + 1 \\
&= 2^{i+2} - 1
\end{aligned}$$

as wanted.

ANSWER TO QUESTION 3.

(a) For convenience we refer to the number of nodes of a binary tree T as the ***size*** of T.

There is exactly one binary tree of height 0; it is a NBBT and has size 1, so $g(0) = 1$. There are three binary trees of height 1. All three of them are NBBTs; two have size 2 and one has size 3. Thus, $g(1) = 2$.

For any $h \geq 2$, let T be a minimum size NBBT of height h. It suffices to prove that the subtrees of T are minimum size NBBTs of height $h-1$ and $h-2$, respectively. Let T_1 and T_2 be the subtrees of T. Without loss of generality, assume that the height of T_1 is greater than or equal to the height of T_2.

(1) T_1 is an NBBT. (Reason: Otherwise, T would not be an NBBT.)

(2) T_1 has height $h-1$. (Reason: T_1 is a tallest subtree of a tree of height h.)

(3) T_1 is a *minimum* size NBBT of height $h-1$. (Reason: By (1) and (2), T_1 is an NBBT of height $h-1$. If it didn't have minimum size, we could replace it in T by a minimum size NBBT of height $h-1$. The resulting tree would be an NBBT of height h and fewer nodes than T, contradicting the definition of T.)

(4) T_2 has height $h-1$ or $h-2$. (Reason: Otherwise the subtrees of the root of T would not have heights that differ by at most one, and T would not be an NBBT.)

(5) T_2 is an NBBT. (Reason: Otherwise, T would not be an NBBT.)

(6) T_2 is a *minimum* size NBBT of height $h-1$ or $h-2$. (Reason: Similar to the argument for (3).)

(7) T_2 is a minimum size NBBT of height $h-2$. (Reason: Otherwise, by the preceding step, T_2 would have to be a minimum size NBBT of height $h-1$. Then the size of T is $2 \cdot g(h-1)+1$. Replace T_2 in T by a minimum size NBBT of height $h-2$. This results in an NBBT T' of height h, whose size is $g(h-1)+g(h-2)+1$. By strict monotonicity of g, $g(h-2) < g(h-1)$. Therefore, the size of T' is less than the size of T, contradicting the definition of T.)

Since T is a minimum size NBBT of height $h-1$, and we have shown (cf. (3) and (7) above) that its two subtrees are minimum size NBBTs of height $h-1$ and $h-2$, it follows that $g(h) = g(h-1)+g(h-2)+1$, as wanted.

(b) Let $P(k)$ be the predicate "$g(h) \geq 1.6^h$". We'll prove that $P(h)$ holds for all $h \geq 0$, by complete induction. Let i be an arbitrary natural number. Assume that $P(j)$ holds for all j such that $0 \leq j < i$. We must show that $P(i)$ holds as well. There are three cases:

CASE 1. $i = 0$. $g(0) = 1$ (by Part (a)), and $1.6^0 = 1$, so $g(0) \geq 1.6^0$; $P(i)$ holds in this case.

CASE 2. $i = 1$. $g(1) = 2$ (by Part (a)), and $1.6^1 = 1.6$, so $g(1) \geq 1.6^1$; $P(i)$ holds in this case.

CASE 3. $i \geq 2$. In this case, $0 \leq i-1 < i$ and $0 \leq i-2 < i$, so $P(i-1)$ and $P(i-2)$ both hold. We have:

$$\begin{aligned}
g(i) &= g(i-1)+g(i-2)+1 && \text{[by Part (a), since } i \geq 2\text{]} \\
&\geq 1.6^{i-1}+1.6^{i-2}+1 && \text{[by induction hypothesis]} \\
&= 1.6^{i-2}(1.6+1)+1 \\
&= 1.6^{i-2} \times 2.6+1 \\
&\geq 1.6^{i-2} \times 1.6^2+1 && \text{[because } 2.6 \geq 1.6^2\text{]} \\
&\geq 1.6^i+1 \\
&\geq 1.6^i
\end{aligned}$$

So, $P(i)$ holds in this case.

(c) We need to find the maximum nonnegative real x so that $x^{h-1} + x^{h-2} + 1 \geq x^h$, for all $h \geq 2$ — i.e.,

$$x^{h-2}(x^2 - x - 1) \leq 1 \qquad \text{for all } h \geq 2$$

From Part (b) we know that $x \geq 1.6$. Since $x > 1$, the factor x^{h-2} in the above equation grows without bound, as h grows. Thus, to keep $x^{h-2}(x^2 - x - 1) \leq 1$ for all values of $h \geq 0$, as required, we must choose x to be the maximum nonnegative real so that $x^2 - x - 1 \leq 0$ (otherwise, the entire $x^{h-2}(x^2 - x - 1)$ will grow without bound as h grows, and we cannot ensure that it is at most 1, for all values of h). From elementary calculus $x^2 - x - 1$ is monotone increasing when $x \geq 1/2$, and we know that $x \geq 1.6$, so $x^2 - x - 1$ is monotone increasing in the range of interest. Therefore, the maximum nonnegative real value of x that ensures $x^2 - x - 1 \leq 0$, is the maximum value of x that makes $x^2 - x - 1 = 0$. By solving this quadratic equation we discover that its positive root is $\phi = (1 + \sqrt{5})/2 = 1.61803\ldots$ — our old friend, the golden ratio. This is the desired value of x.

ANSWER TO QUESTION 4. Let $d = 28$.

(a) Let $P(n)$ be the predicate "there are natural numbers a and b such that $n = 5 \cdot a + 8 \cdot b$" (in other words, $P(n)$ asserts that it is possible to form postage n using the available denominations of stamps: a represents the number of stamps of value 5, and b the number of stamps of value 8). We use induction to prove that $P(n)$ holds, for all $n \geq 28$.

Basis: $n = 28$. $28 = 5 \cdot 4 + 8 \cdot 1$, so $P(28)$ holds (take $a = 4$ and $b = 1$).

Induction Step: Let i be an arbitrary integer $i \geq 28$. Assume that $P(i)$ holds; i.e., for some natural numbers a and b, $i = 5 \cdot a + 8 \cdot b$. We must prove that $P(i+1)$ holds as well; i.e., for some natural numbers a' and b', $i + 1 = 5 \cdot a' + 8 \cdot b'$.

Case 1. $a \geq 3$. Let $a' = a - 3$ and $b' = b + 2$. Since $a \geq 3$, a' and b' are both natural numbers. Straightforward arithmetic and the induction hypothesis that $i = 5 \cdot a + 8 \cdot b$ show that $i + 1 = 5 \cdot a' + 8 \cdot b'$. (Intuitively, in this case we exchange 3 stamps for value 5 with 2 stamps of value 8, for an overall gain of 1.)

Case 2. $a \leq 2$. Then $b \geq 3$ (otherwise $i = 5 \cdot a + 8 \cdot b \leq 5 \cdot 2 + 8 \cdot 2 \leq 26$, contradicting $i \geq 28$). Let $b' = b - 3$ and $a' = a + 5$. Straightforward arithmetic and the induction hypothesis shows that $i + 1 = 5 \cdot a' + 8 \cdot b'$. (Intuitively, in this case we exchange 3 stamps for value 8 with 5 stamps of value 5, for an overall gain of 1.)

(b) Since 5 does not divide any of

$$27 - 0 \cdot 8 = 27, \quad 27 - 1 \cdot 8 = 19, \quad 27 - 2 \cdot 8 = 11, \quad \text{or} \quad 27 - 3 \cdot 8 = 3$$

we cannot form postage 27 using zero, one, two, or three stamps of value 8 and any number of stamps of value 5. On the other hand, we cannot form postage 27 using more than three stamps of value 8 (and any number of stamps of value 5), because the stamps of value 8 alone already make postage that exceeds 27. Thus, there is no combination of stamps of value 8 and value 5 that gives postage 27.

ANSWER TO QUESTION 5. Let $P(k)$ be the statement "if the k-th iteration of the loop occurs, then y_k is a nonnegative multiple of 3". Using induction, we'll prove that $P(k)$ is an invariant for the loop.

BASIS: $y_0 = 3n$, where (by assumption) n is a natural number, so y_0 is a nonnegative multiple of 3.
INDUCTION STEP: Let i be an arbitrary natural number. Suppose that $P(i)$ holds. We'll show that $P(i+1)$ holds as well. There are two cases:

CASE 1. $y_i \bmod 4 = 0$, i.e., y_i is a multiple of 4. Since, by induction hypothesis, y_i is also a nonnegative multiple of 3, it follows that y_i is a nonnegative multiple of 12. Thus, $y_i = 12k$, for some integer $k \geq 0$. Furthermore, $y_i \neq 0$ (otherwise, by the exit condition of the loop, there would be no $(i+1)$-st iteration). Thus $k \geq 1$.

By the program, $y_{i+1} = y_i - 6$, and therefore $y_{i+1} = 12k - 6 = 3(4k-2)$. Since $k \geq 1$, $4k - 2 \geq 0$, so y_{i+1} is a nonnegative multiple of 3.

CASE 2. $y_i \bmod 4 \neq 0$. In this case, by induction hypothesis y_i is a nonnegative multiple of 3, i.e., $y_i = 3k$, for some integer $k \geq 0$. By reasoning as in the previous case, we can conclude that $k \geq 1$.

By the program, $y_{i+1} = y_i - 3$, and therefore $y_{i+1} = 3k - 3 = 3(k-1)$. Since $k \geq 1$, $k - 1 \geq 0$, so y_{i+1} is a nonnegative multiple of 3.

This completes the proof that $P(k)$ is an invariant for the loop.

Now consider the sequence $\langle y_0, y_1, y_2, \ldots \rangle$, for each y_k such that y_k is defined (i.e., a k-th iteration of the loop exists). For any $k \geq 0$, if y_{k+1} is defined, either $y_{k+1} = y_k - 6$ or $y_{k+1} = y_k - 3$; in any case $y_{k+1} < y_k$. By the invariant established above, each y_k is a nonnegative integer. (Note that to prove the invariant we had to strengthen the desired statement and prove that y_k is not only a nonnegative integer but, in fact, a nonnegative multiple of 3.)

Therefore the sequence $\langle y_0, y_1, y_2, \ldots \rangle$ is a *decreasing* sequence of *natural numbers*, and by the Well-Ordering Principle it is finite. Hence, y_k is only defined for finitely many values of k, and so the loop terminates.
