Great Ideas in Computing

University of Toronto CSC196
Fall 2023

Week 12: December 4 - December 6 (2023)

Announcements

Announcements

- As you know, this is the last week of classes for the fall term. We will not avail ourselves of the makeup Monday class as I am sure everyone would rather have the time to study for exams.
- Please promptly submit any regrade requests for quiz 2.
- I have some quizzes that were not picked up last Wednesday. Any unclaimed quizzes after the last class this Wednesday (December 6) will be given to the undergraduate office.
- The final assignment is due December 6 at 9AM. There will not be any extensions past this new due date. I am sure that Aniket will grade them as fast as possible. After the Assignments and grades are posted on Quercus, I will calculate the participation grade and then calculate final grades. I will post (on Quercus) these grades before submitting. Please notify (by email) within two days if you notice any errors. I am supposed to submit final grades within a week of the end of classes for courses not having a final exam.

This weeks agenda

- The topic this week is social networks.
- In particular, the main theme will be how graph structure can reveal personal and individual information as well as communities. In particular, we will discuss
- Floretine marriages and "centrality". Why were the Medici's so influential?
- The Bearman et al study of romantic relations in a US high school which we mentioned briefly before.
- The Backstrom and Kleinberg method for discovering *the* romantic relation in a subgraph of facebook.
- Bearman and Moody discussion of low triadic closure
- Modelling and understanding the small worlds phenomena. The Watts-Strogaatz, and Kleinberg models and analysis.
- More realistic georgraphic models.
- Extending geographic distance to social distance.

Social networks

A social network is a network $G = (V, E)$ where the nodes in V are people or organizations. Social networks can be undirected or directed networks.

The edges can be relations between people (e.g. friendship) or membership of an individual in an organization.

Social networks can be of any size (e.g., a small network like the Karate Club on slide 16, later in todays slides) or enormous networks like Facebook and Twitter. We usually think of Facebook as an undirected graph (where *friendship* is an undirected edge) and Twitter as a directed graph (i.e., where *follows* is a directed edge).

Understanding how networks evolve, the resulting structure of social networks, and computational aspects for dealing with large networks is an active field of study in CS as well as in sociology, political science, economics, epidemiology, and any field that studies human behaviour. J. Kleinberg's 2000 analysis with regard to the six degrees of separation phenomena is an early result that sparked interest in algorithmic aspects of social networks.

The computational challenge presented by super large networks

The size of some modern networks such as the web and social networks such as Facebook are at an unprecedented scale.

As of Februay, 2022, xThe average facebook user has about 155 friends which then implies about $2.9 \cdot \frac{155}{2} \approx 200$ billion edges. It is interesting to note that 90% of daily active users are outside USA and Canada. See https://www.omnicoreagency.com/facebook-statistics/
for lots of interesting demographic and other facts about Facebook.

What does this imply for the complexity of algorithms involving such super large networks?

Linear is the new exponential

In complexity theory (e.g. in the P vs NP) we say (as an abstraction) that polynomial time algorithms are "efficient" and "exponential time" is infeasible. There are, of course, exceptions but as an abstraction this has led to invaluable fundamental insights.

As problem instances have grown, there was a common saying that "quadratic (time) is the new exponential".

But with the emergence of networks such as the web graph and the Facebook network, we might now say that "linear is the new exponential" when it comes to extracting even the most basic facts about these networks. For example, how do we even estimate the average node degree in a giant network?

There are many facts about large networks that we would like to extract from the network. For example, how do we find "influential" or "interesting nodes" in a social network?

Sublinear time algorithms

What is sublinear time?

In general when we measure complexity, we do so as a funtion of the input/output size. For graphs $G = (V, E)$, the size of the input is usually the number of edges E. (An exception is that when the graph is presented say as an adjacency matrix, the size is n^2 where $n = |V|$.)

Since our interest is in massive information and social networks, we consider sparse graphs (e.g. average constant degree) so that $|E| = O(|V|)$ and hence we will mean sublinear time as a function of n (equivalently $m = |E|$). The desired goal will be time bounds of the form $O(n^\alpha)$ with $\alpha < 1$ and in some cases maybe even $O(\log n)$ or $polylog(n)$.

Given that optimal algorithms for almost any graph property will depend on the entire graph, we will have to settle for approximations to an optimum solution. Furthermore, we will need to sample the graph so as to avoid having to consider all nodes and edges. And we will need a way to efficiently access these massive graphs,

Coping with massive social graphs continued

One way to help coping with massive networks is to hope to utilize some substantial amount of parallelism. There is an area of current research concerning massive parallel computation (MPC) models where (in principle) we can achieve sublinear time by distributing computation amongst a large (i.e., conceptually a non constant) number of processors.

But even if we could muster and organize thousands of machines, we will still need random samplng, approximation, and have highly efficient "local information algorithms" (e.g., where say each processor is responsible for some nodes and learns about its local neighbourhood).

Finally, in addition to random sampling and parallelism, we will have to hope that social networks have some nice structural properties that can be exploited to as to avoid complexity barriers that exist for arbitrary (even sparse) graphs. These complexity barriers are hopefully clear from our discussion of complexity theory, *NP completeness* and *NP hardness*.

Preferential attachment models

Preferential attachnment models (also called "rich get richer" models) are probabilistic generative models explaining how various networks can be generated. Namely, after starting with some small graph, when we add a new node u, we create a number of links between u to some number m of randomly chosen existing nodes $v_1, v_2, \ldots, v_m$. The probability of choosing a v_i is proportional to the current degree of v_i.

These models have been used to help explain the structure of the web as well as social networks. Furthermore, networks generated by such a process have some nice structural properties allowing for substantially more efficient algorithms than one can obtain for arbitrary graphs.

For such models, there are both provable analytic results as well as experimental evidence on synthetic and real networks that support provable results that follow from the model. (Remember, a model is just a model and is not "reality"; as models are implifications of real networks, they may not account for many aspects in a real network. For example, in this basic model, all the edges for a new node are set upon arrival.)

Consequences for networks generated by a preferential attachment process

There are many properties, believed and sometimees proven. about preferential attachment network models that do not hold for uniformly generated random graphs (e.g., if we create random sparse graphs with constant average degree by choosing each possible edge with say probability proportional to $\frac{1}{n}$).

One of the most interesting and consequential proerties is that vertex degrees satisfy a *power law distribution* in expectation. Specifically, the expectation fraction $P(d)$ of nodes whose degree is d is proportional to $d^{-\gamma}$ for some $\gamma \geq 1$. Such a distribution is said to have a *fat tail*.

In a uniformaly random sparse graph (with average degree d_{avg}), with high probability , the fraction of nodes having a large degree $d > d_{avg}$ is proportional to c^{-d} for some $c > 1$.

The Barabasi and Albert preferential model

Barabasi and Albert [1999] specified a particular preferential attachment model and conjectured that the vertex degrees satisfy a power law in which the fraction of nodes having degree d is proportional to d^{-3}.

They obtained $\gamma \approx 2.9$ by experiments and gave a simple heuristic argument suggesting that $\gamma = 3$. That is, $P(d)$ is proportional to d^{-3}

Bollobas et al [2001] prove a result corresponding to this conjectured power law. Namely, they show for all $d \leq n^{1/15}$ that the *expected* degree distribution is a power law distribution with $\gamma = 3$ asymptotically (with n) where n is the number of vertices.

Note: It is known that an actual realized distribution may be far from its expectation, However, for small degree values, the degree distribution is close to expectation.

When we say that a distribution $P(d)$ is a power law distribtion this is often meant to be a "with high probability" whereas many results for networks generated by a preferential attachment process the power law is usually only in expectation.

Proven or observed properties of nodes in a social network generated by preferential attachment models

In addition to the power law phenomena suggesting many nodes with high degree, other properies of social networks have been observed such as a relatively large number of nodes u having some or all of properties such as the following: .

- high clustering coefficient defined as : $\frac{(u,v),(u,w),(v,w)\in E}{(u,v),(u,w)\in E}$. That is, mutual friends of u are likely to be friends.
- high centrality ; e,g, nodes on many pairs of shortest paths.

Brautbar and Kearns refer to such nodes (as above) as "interesting indiviudals" and these individuals might be candidates for being "highly influential individuals". Bonato et al [2015] refers to such nodes as the *elites* of a social network.

Other proven or observed properties of networks generated by preferentical attachment models

- correlation between the degree of a node u and the degrees of the neighboring nodes.
- the graph has small diameter; suggesting "6 degrees of separation phenomena"
- relatively large dense subgraph communities.
- rapid mixing (for random walks to approach stationary distribution)
- relatively small (almost) *dominating sets*. What do we mean by "almost"?

On my spring 2020 CSC303 web page, I posted a paper by Avin et al (2018) that shows that preferential attachment is the *only* "rational choice" for players (people) playing a simple natural network formation game. It is the rational choice in the sense that the strategy of the players will lead to a unique equilibrium (i.e. no player will want to deviate assumming other players do not deviate). For those intersted, I have posted (in my CSC303 webpage) a number of other papers on elites in a social network and preferential attachment.

End of Monday, December 4 class

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On Wednesday, we will quickly present a number of studies that illustrate the use of graph structure in obtaining information in a social-network. This is just meant to generate interest in the computational study of social networks. We will quickly consider:

- The centrality and influence of a node.
- Detecting communities and influential nodes.
- Detecting the romantic relation in a Facebook network
- The importance of triadic closure and low clustering coefficient.
- The six degrees of separation phenomena

Florentine marriages and "centrality"

- Medici connected to more families, but not by much
- More importantly: lie between most pairs of families
  - shortest paths between two families: coordination, communication
  - Medici lie on 52% of all shortest paths; Guadagni 25%; Strozzi 10%

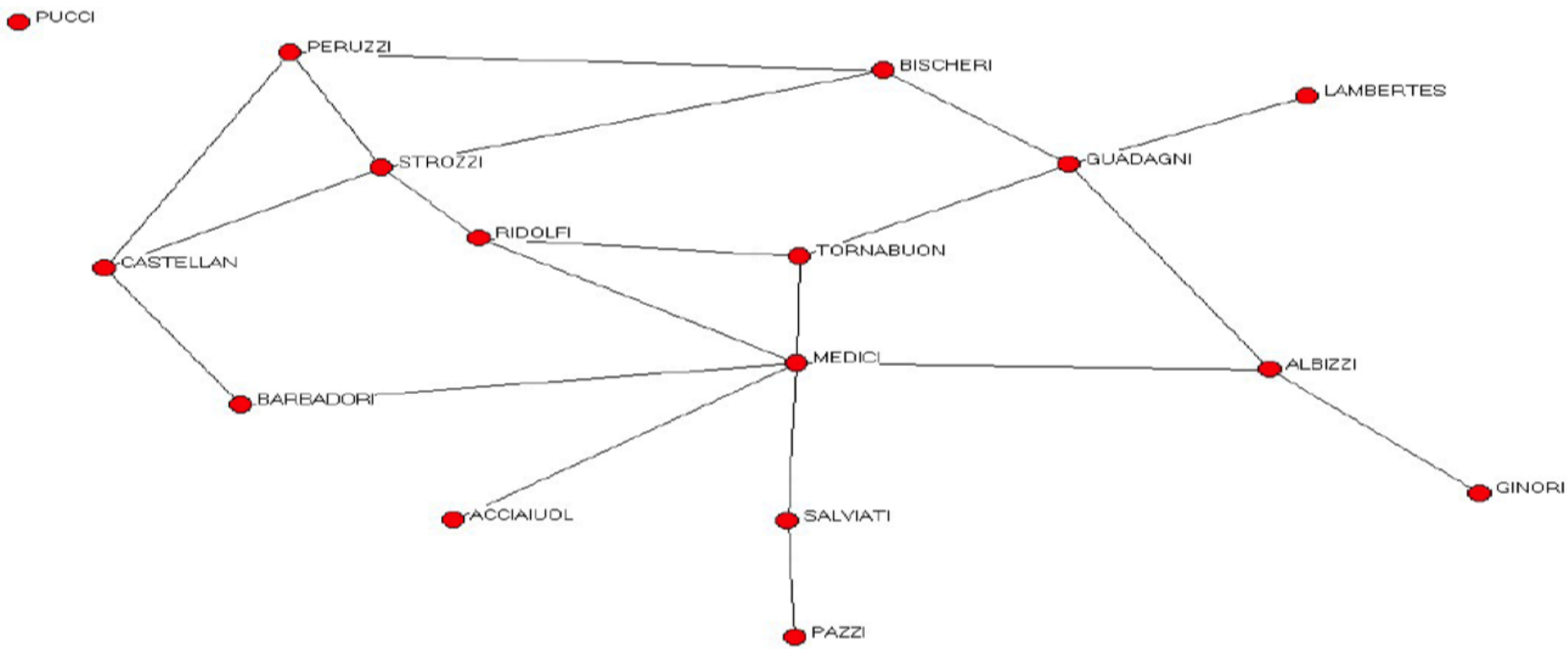

Figure: see [Jackson, Ch 1]

Example of communities and central nodes

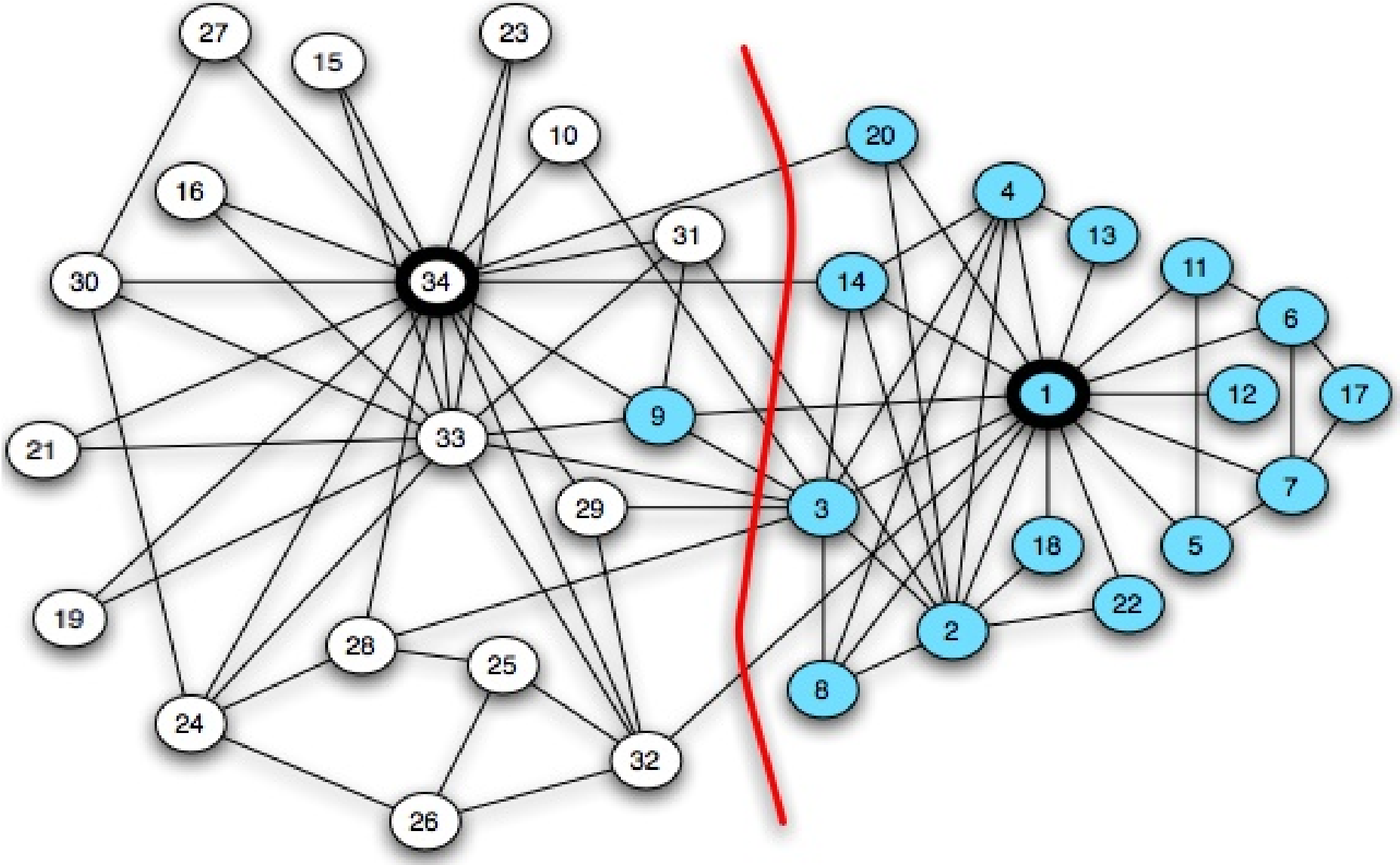

Figure: Zachary Karate Club [1977]. The figure illustates a *min cut* partitiuoning the network. Also not the centrality of nodes 1 and 34.

How graph structure can reveal personal information: Detecting the romantic relation in Facebook

- There is an interesting paper by Backstrom and Kleinberg (http://arxiv.org/abs/1310.6753) on detecting "the" romantic relation in a subgraph of facebook users who specify that they are in such a relationship.
- Backstrom anbd Kleinberg construct two datasets of randomly sampled Facebook users: (i) an extended data set consisting of 1.3 million users declaring a spouse or relationship partner, each with between 50 and 2000 friends and (ii) a smaller data set extracted from neighbourhoods of the above data set (used for the more computationally demanding experimental studies).
- The main experimental results are nearly identical for both data sets.

Detecting the romantic relation (continued)

- They consider various graph strucutral features of edges, including
  1. the *embeddedness* of an edge (A, B) which is the number of mutual friends of A and B.
  2. various forms of a new *dispersion* measure of an edge (A, B) where high dispersion intuitively means that the mutual neighbours of A and B are not "well-connected" to each other (in the graph without A and B).
  3. One definition of dispersion given in the paper is the number of pairs (s, t) of mutual friends of u and v such that $(s, t) \notin E$ and s, t have no common neighbours except for u and v.
- They also consider various "interaction features" including
  1. the number of photos in which both A and B appear.
  2. the number of profile views within the last 90 days.

Embeddedness and disperison example from paper

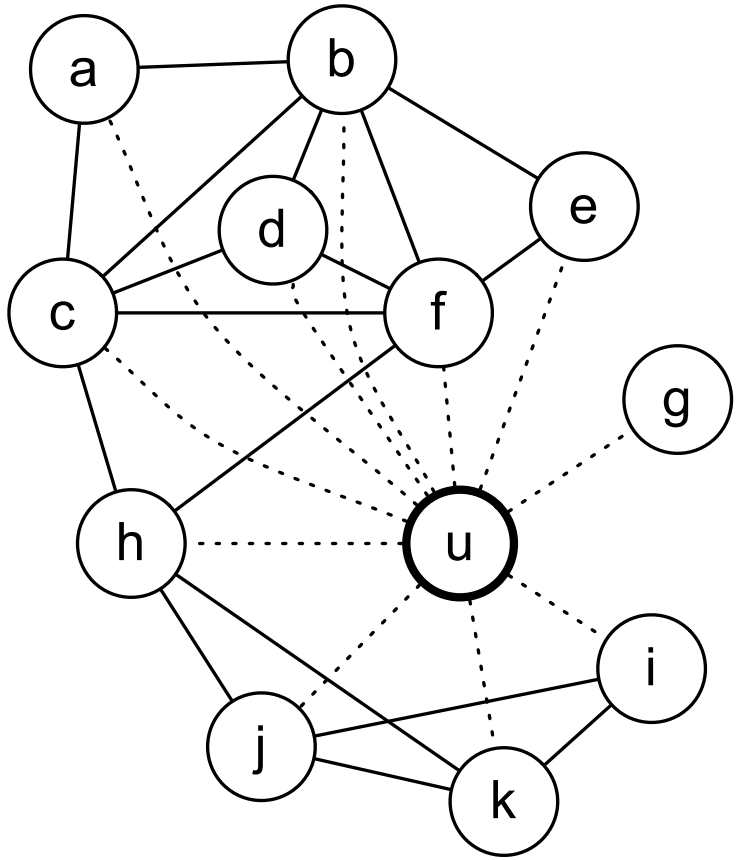

Figure 2. A synthetic example network neighborhood for a user u; the links from u to b, c, and f all have embeddedness 5 (the highest value in this neighborhood), whereas the link from u to h has an embeddedness of 4. On the other hand, nodes u and h are the unique pair of intermediaries from the nodes c and f to the nodes j and k; the u-h link has greater dispersion than the links from u to b, c, and f.

Qualitative results from Backstrom and Kleinberg

- The goal is to predict (for each user in the data set) which of their friendship edges is the romantic relation. Note that each user has between 50 and 2000 friends and assuming say a median of 200 users, a random guess would have prediction accuracy of $1/200 = .5\%$

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- For most measures, performance is better for male users and also better for data when restricted to marriage as the relationship.
- By combining many features, structural and interaction, the best performance is achieved using machine learning classification algorithms based on these many features.
- There are a number of other interesting observations but for me the main result is the predictive power provided by graph structure although there will generally be a limit to what can be learned solely from graph structure.

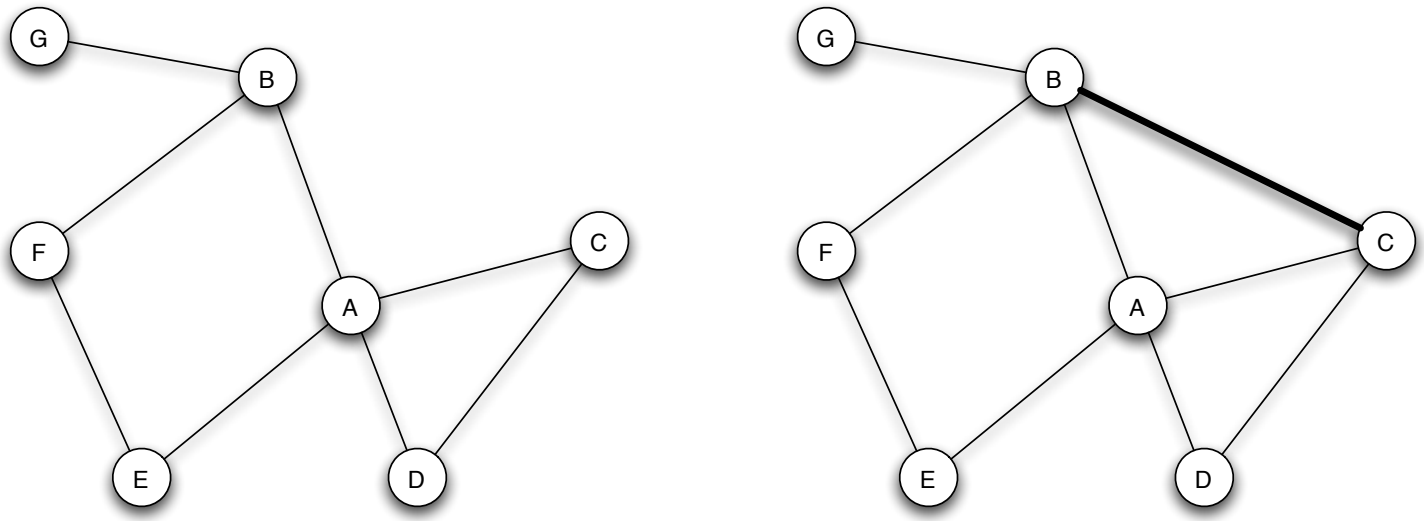
Some experimental results for the fraction of correct predictions

Recall that we argue that the fraction might be .005 when randomly choosing an edge. Do you find anything surprising?

type	embed	rec.disp.	photo	prof.view.
all	0.247	0.506	0.415	0.301
married	0.321	0.607	0.449	0.210
married (fem)	0.296	0.551	0.391	0.202
married (male)	0.347	0.667	0.511	0.220
engaged	0.179	0.446	0.442	0.391
engaged (fem)	0.171	0.399	0.386	0.401
engaged (male)	0.185	0.490	0.495	0.381
relationship	0.132	0.344	0.347	0.441
relationship (fem)	0.139	0.316	0.290	0.467
relationship (male)	0.125	0.369	0.399	0.418

type	max. struct.	max. inter.	all. struct.	all. inter.	comb.
all	0.506	0.415	0.531	0.560	0.705
married	0.607	0.449	0.624	0.526	0.716
engaged	0.446	0.442	0.472	0.615	0.708
relationship	0.344	0.441	0.377	0.605	0.682

Triadic closure (undirected graphs)



(a) *Before B-C edge forms.* (b) *After B-C edge forms.*

Figure: The formation of the edge between B and C illustrates the effects of triadic closure, since they have a common neighbor A. [E&K Figure 3.1]

- Triadic closure: mutual "friends" of say A are more likely (than "normally") to become friends over time.
- How do we measure the extent to which triadic closure is occurring?
- How can we know why a new friendship tie is formed? (Friendship ties can range from "just knowing someone" to "a true friendship" .)

Measuring the extent of triadic closure

- The clustering coefficient of a node A is a way to measure (over time) the extent of triadic closure (perhaps without understanding why it is occurring).

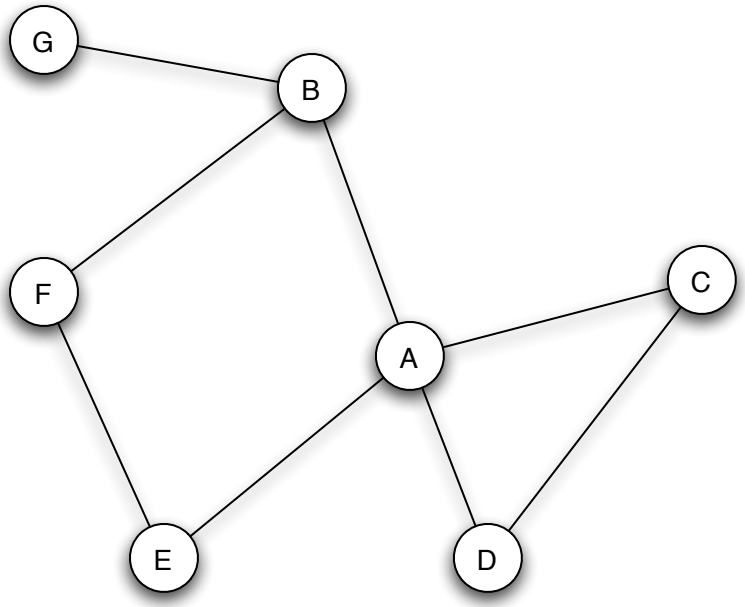
- Let E be the set of an undirected edges of a network graph. (Forgive the abuse of notation where in the previous and next slide E is a node name.) For a node A, the clustering coefficient is the following ratio:

$$\frac{\left|\left\{(B,C) \in E : (B,A) \in E \text{ and } (C,A) \in E\right\}\right|}{\left|\left\{\{B,C\} : (B,A) \in E \text{ and } (C,A) \in E\right\}\right|}$$

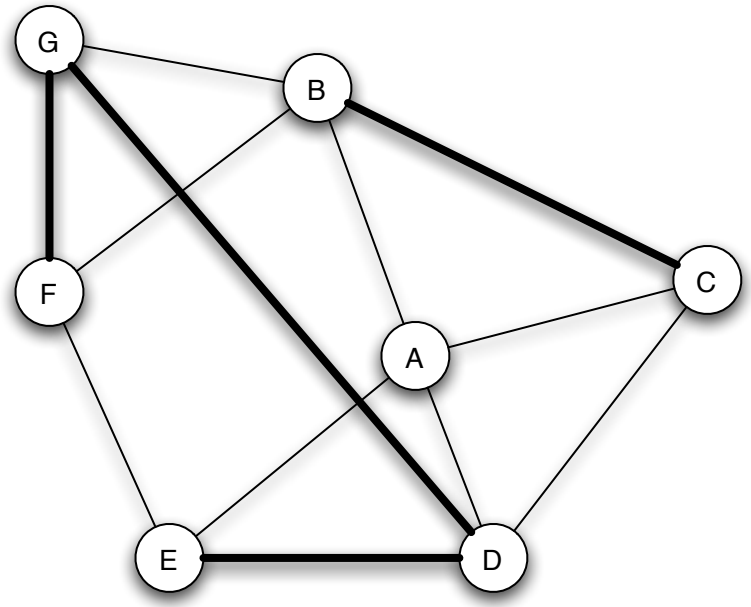
- The numerator is the number of all edges (B, C) in the network such that B and C are adjacent to (i.e. mutual friends of) A.

- The denominator is the total number of all unordered pairs $\{B, C\}$ such that B and C are adjacent to A.

Example of clustering coefficient



(a) *Before new edges form.*



(b) *After new edges form.*

- The clustering coefficient of node A in Fig. (a) is $1/6$ (since there is only the single edge (C, D) among the six pairs of friends: $\{B, C\}$, $\{B, D\}$, $\{B, E\}$, $\{C, D\}$, $\{C, E\}$, and $\{D, E\}$). We sometimes refer to a pair of adjacent edges like $(A, B), (A, C)$ as an "open triangle" if (B, C) does not exist.
- The clustering coefficient of node A in Fig. (b) increased to $1/2$ (because there are three edges (B, C), (C, D), and (D, E)).

Interpreting triadic closure

- Does a low clustering coefficient suggest anything?

Interpreting triadic closure

- Does a low clustering coefficient suggest anything?

- Bearman and Moody [2004] reported finding that a low clustering coefficient amongst teenage girls implies a higher probability of contemplating suicide (compared to those with high clustering coeficient). Note:The value of the clustering coefficient is also referred to as the *intransitivity coefficient*.

- They report that " Social network effects for girls overwhelmed other variables in the model and appeared to play an unusually significant role in adolescent female suicidality. These variables did not have a significant impact on the odds of suicidal ideation among boys. "

How can we understand these findings?

Bearman and Moody study continued

- Triadic closure (or lack thereof) can provide some plausible explanation.

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- Triadic closure (or lack thereof) can provide some plausible explanation.
  Increased opportunity, trust, incentive ; it can be awkward to have friends (especially good friends with strong ties) who are not themselves friends.
  As far as I can tell, no conclusions are being made about why there is such a difference in gender results.

The study by Bearman and Moody is quite careful in terms of identifying many possible factors relating to suicidal thoughts. Clearly there are many factors involved but the fact that network structure is identified as such an important factor is striking.

Bearman and Moody factors relating to suicidal thoughts

TABLE 3—Logistic Regression of Suicide Attempts, Among Adolescents With Suicidal Ideation, on Individual, School, Family and Network Characteristics

	Suicide Attempts, OR (95% CI)	
	Males	Females
Demographic		
Age	0.956 (0.808, 1.131)	0.920 (0.810, 1.046)
Race/ethnicity		
Black	0.872 (0.414, 1.839)	1.086 (0.680, 1.736)
Other	1.069 (0.662, 1.728)	1.134 (0.810, 1.586)
Socioeconomic status	0.948 (0.872, 1.031)	1.008 (0.951, 1.069)
School and community		
Junior high school	1.588 (0.793, 3.180)	1.271 (0.811, 1.993)
Relative density	0.049 (0.005, 0.521)	0.415 (0.086, 1.996)
Plays team sport	0.985 (0.633, 1.532)	1.020 (0.763, 1.364)
Attachment to school	1.079 (0.823, 1.414)	1.066 (0.920, 1.235)
Religion		
Church attendance	0.975 (0.635, 1.496)	0.818 (0.618, 1.082)
Family and household		
Parental distance	0.925 (0.681, 1.256)	0.955 (0.801, 1.139)
Social closure	1.004 (0.775, 1.299)	0.933 (0.781, 1.115)
Stepfamily	1.058 (0.617, 1.814)	1.368 (0.967, 1.935)
Single-parent household	1.142 (0.698, 1.866)	1.117 (0.800, 1.560)
Gun in household	1.599 (1.042, 2.455)	1.094 (0.800, 1.494)
Family member attempted suicide	1.712 (0.930, 3.150)	1.067 (0.689, 1.651)
Network		
Isolation	0.767 (0.159, 3.707)	1.187 (0.380, 3.708)
Intransitivity index	0.444 (0.095, 2.085)	1.076 (0.373, 3.103)
Friend attempted suicide	1.710 (1.095, 2.671)	1.663 (1.253, 2.207)
Trouble with people	1.107 (0.902, 1.357)	1.119 (0.976, 1.284)
Personal characteristics		
Depression	1.160 (0.960, 1.402)	1.130 (0.997, 1.281)
Self-esteem	1.056 (0.777, 1.434)	0.798 (0.677, 0.942)
Drunkenness frequency	1.124 (0.962, 1.312)	1.235 (1.115, 1.368)
Grade point average	0.913 (0.715, 1.166)	0.926 (0.781, 1.097)
Sexually experienced	1.323 (0.796, 2.198)	1.393 (0.990, 1.961)
Homosexual attraction	1.709 (0.921, 3.169)	1.248 (0.796, 1.956)
Forced sexual relations		1.081 (0.725, 1.613)
No. of fights	0.966 (0.770, 1.213)	1.135 (0.983, 1.310)
Body mass index	0.981 (0.933, 1.032)	1.014 (0.982, 1.047)
Response profile (n = 1/n = 0)	139/493	353/761
F statistic	1.84 ($P = .0170$)	2.88 ($P < .0001$)

Note. OR = odds ratio; CI = confidence interval. Logistic regressions; standard errors corrected for sample clustering and stratification on the basis of region, ethnic mix, and school type and size.

The Small World Phenomena

I already mentioned the small worlds phenomena. A mathematical explanation of this phenomiena (expecially how one hones in on a target recipient) was given by J. Kleinberg in a network formation model that explicitly forces a power law property.

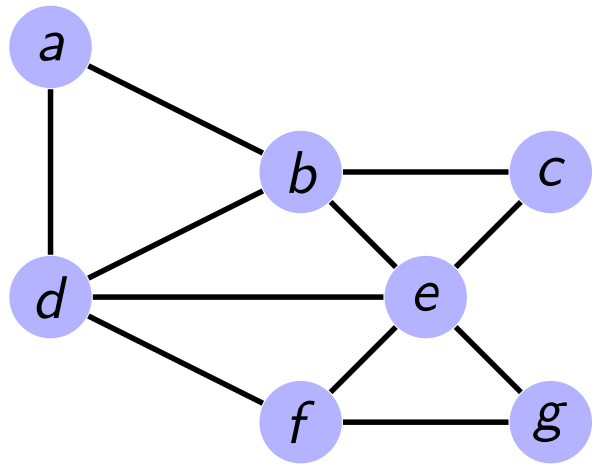
The small world phenomena suggests that in a connected social network any two individuals are likely to be connected (i.e. know each other indirectly) by a short path. Moreover, such a path can be found in a decentralized manner

In Milgram's 1967 small world experiment, he asked random people in Omaha Nebraska to forward a letter to a specified individual in a suburb of Boston which became the origin of the idea of six degrees of separation.

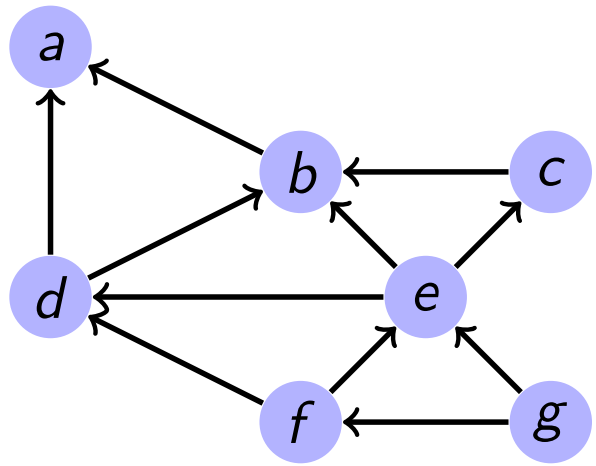
Appendix: Network (graph) definitions and examples

Graphs come in two varieties

1. undirected graphs ("graph" usually means an undirected graph.)



2. directed graphs (often called di-graphs).



Visualizing Networks as Graphs

- nodes: entities (people, countries, companies, organizations, . . .)
- links (may be directed or weighted): relationship between entities
  - friendship, classmates, did business together, viewed the same web pages, . . .
  - membership in a club, class, political party, . . .

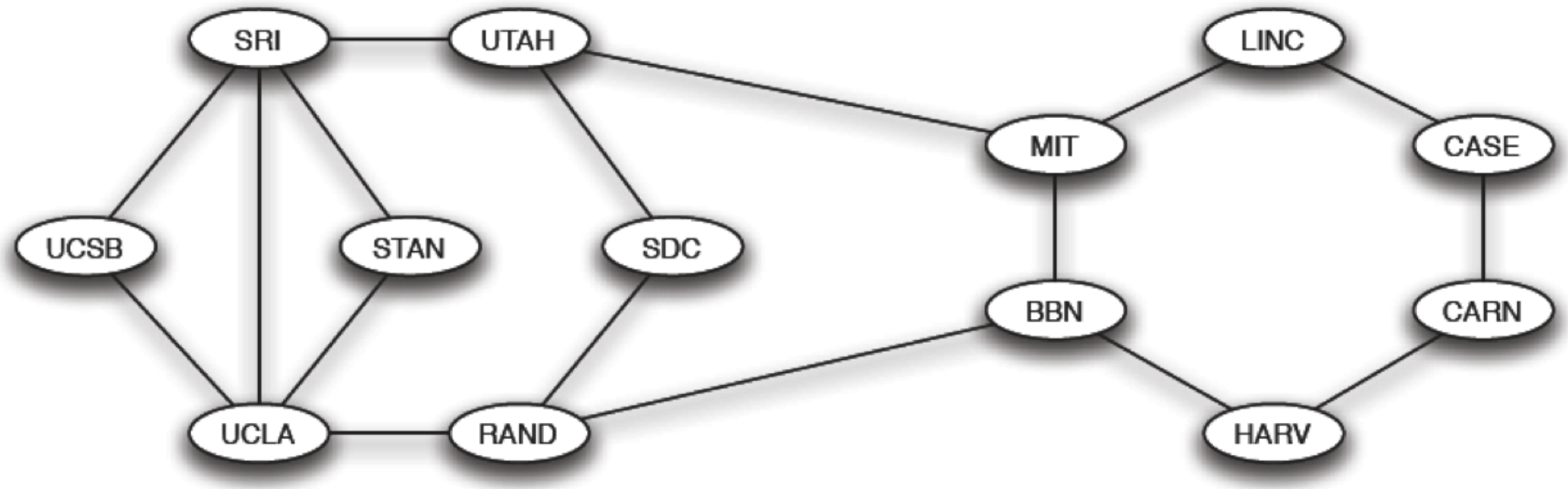


Figure: Internet: Dec. 1970 [E&K, Ch.2]

Adjacency matrix for graph induced by eastern sites) in 1970 internet graph: another way to represent a graph

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- This node induced subgraph (for the sites MIT = 1, LINC = 2, CASE = 3, CARN = 4, HARV = 5, BBN = 6) is a 6 node regular graph of degree 2. It is a simple graph in that there are no self-loops or multiple edges.
- Note that the adjacency matrix of an (undirected) simple graph is a symmetric matrix (i.e. $A_{i,j} = A_{j,i}$) with $\{0,1\}$ entries.
- To specify distances, we would need to give weights to the edges to represent the distances.

The matrix A^2 where $A = A(G)$

Consider squaring the previous matrix $A = A(G)$. That is, $A^2 = A * A$.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Draw a visualization of the graph represented by A^2. If we let $c_{i,j}$ be the i, j entry in A^2, can you desribe the meaning of $c_{i,j}$?

The matrix $B = A + I$

Consider the 6×6 identity matrix $I = (\iota_{i,j})$. That is, $\iota_{i,i} = 1$ for $1 \leq i \leq 6$ and $\iota_{i,j} = 0$ for $1 \leq i, j \leq 6$ and $i \neq j$.

Let $B = A + I$ (as above). That is, $b_{i,j} = a_{i,j} + \iota_{i,j}$ for all i, j. We have

$$B(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that now the matrix B has self loops and hence is not a simple graph.

Breadth first search and path lengths [E&K, Fig 2.8]

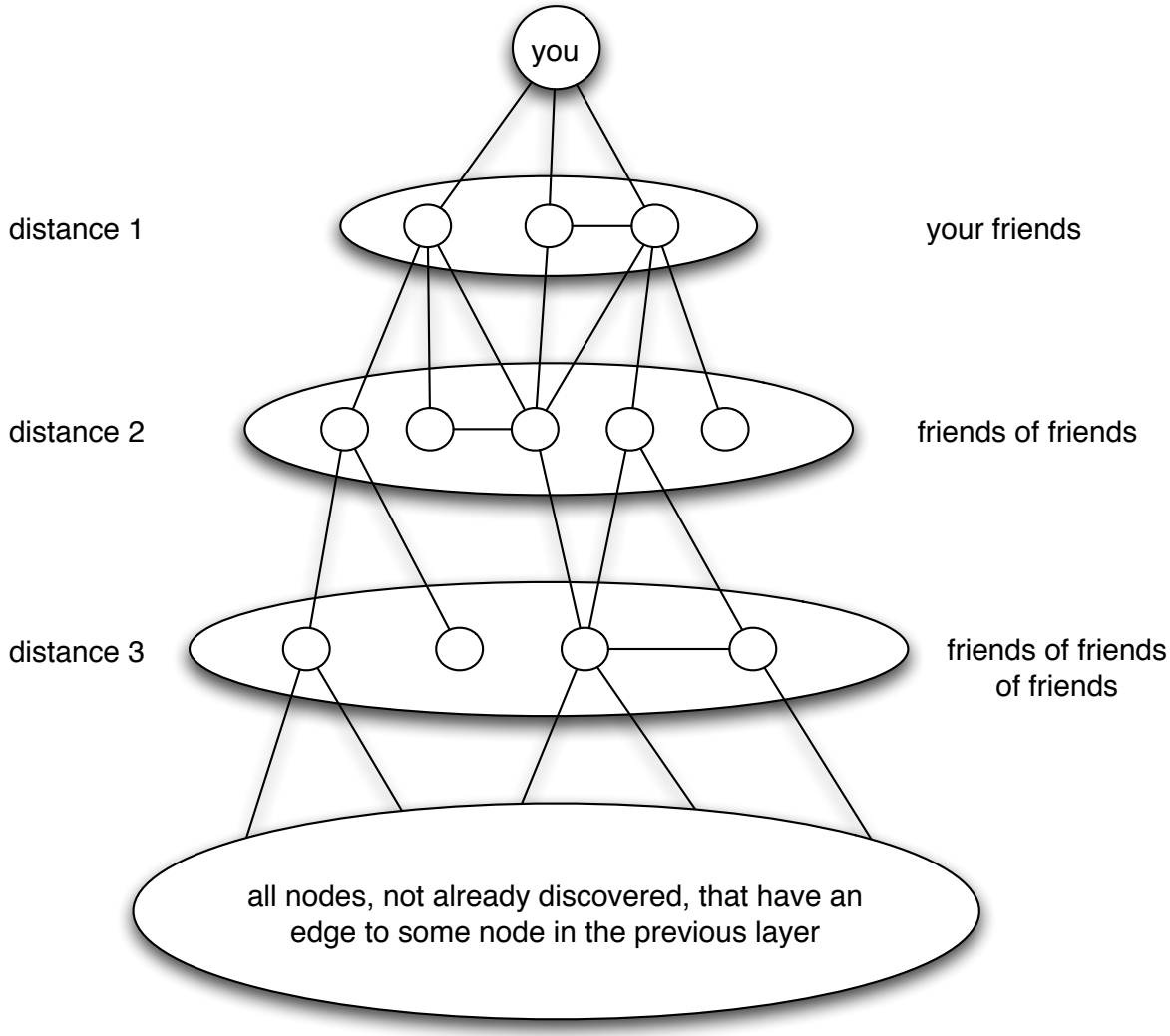


Figure: Breadth-first search discovers distances to nodes one "layer" at a time. Each layer is built of nodes adjacent to at least one node in the previous layer.

Analogous concepts for directed graphs

- We use the same notation for directed graphs, i.e. denoting a di-graph as $G = (V, E)$, where now the edges in E are directed.

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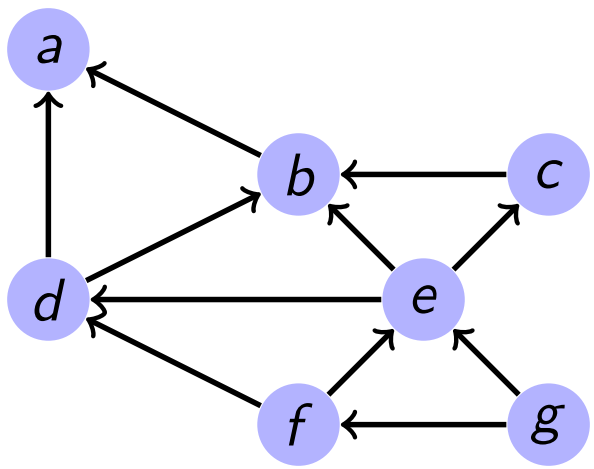
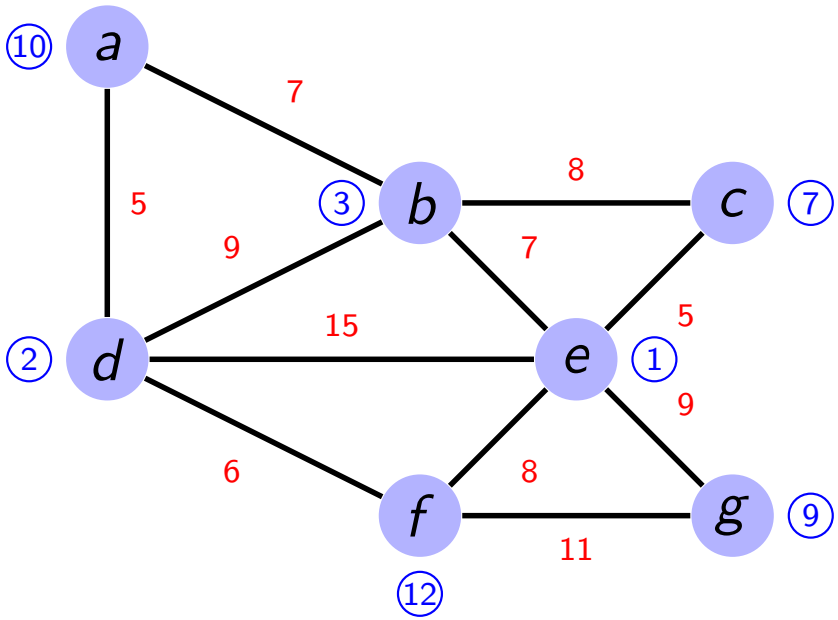
  - However, it is usually clear from context if we are discussing undirected or directed graphs and in both cases most people just write (u, v).
- We now have directed paths and directed cycles. Instead of connected components, we have strongly connected components.



Weighted graphs

- We will often consider weighted graphs. Lets consider a (directed or undirected) graph $G = (V, E)$. Example:



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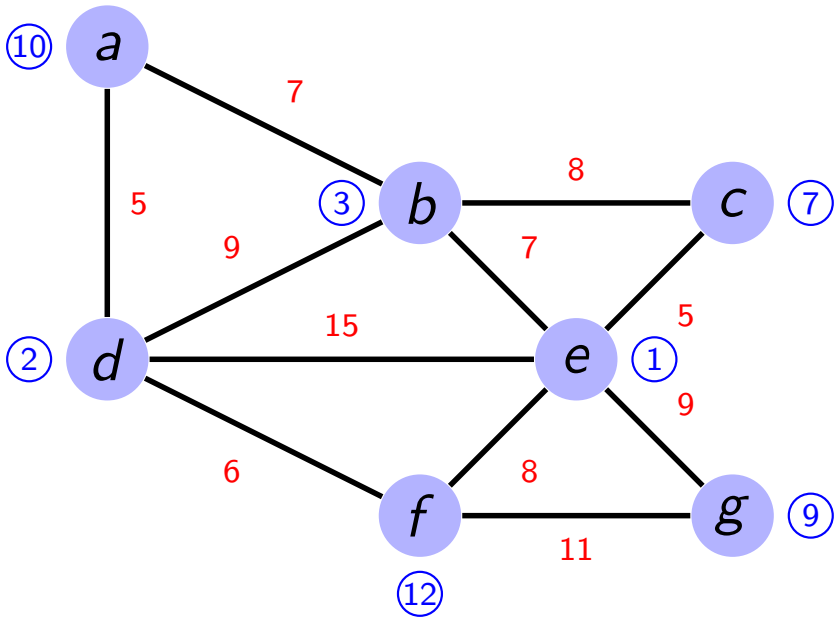

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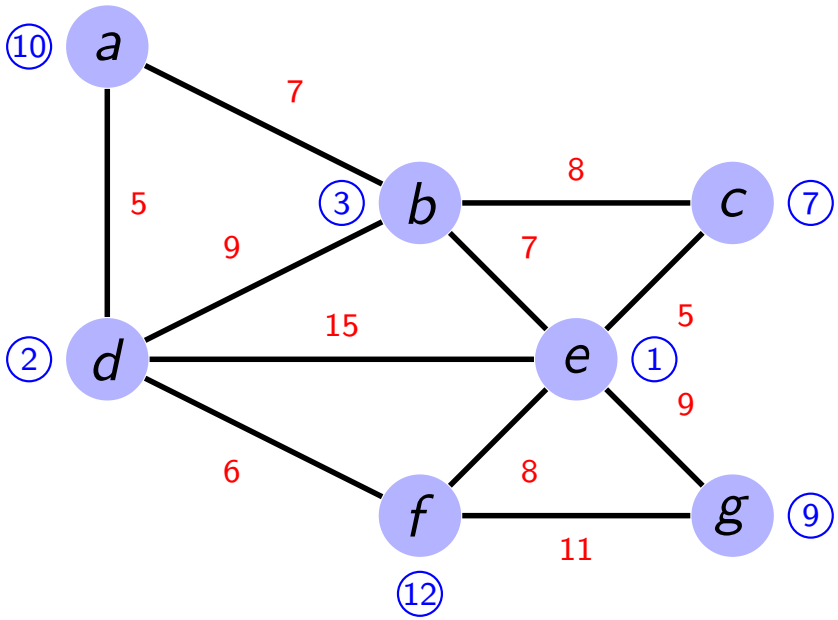

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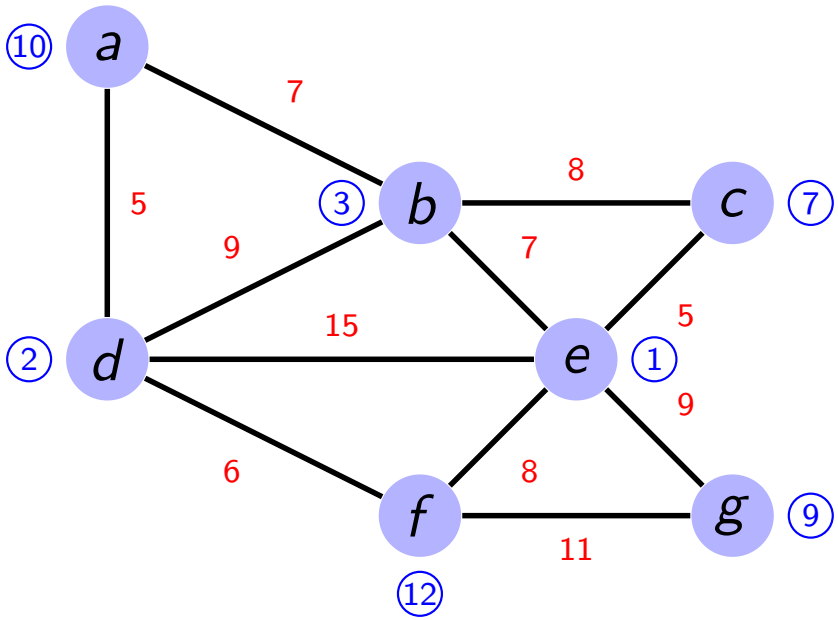

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- For example, in a social network whose nodes represent people, the weight $w(v)$ of node v might indicate the importance of this person.
- The weight $w(e)$ of edge e might reflect the strength of a friendship.

Edge weighted graphs

- When considering edge weighted graphs, we often have edge weights $w(e) = w(u, v)$ which are non negative (with $w(e) = 0$ or $w(e) = \infty$ meaning no edge depending on the context).

- In some cases, weights can be either positive or negative. A positive (resp. negative) weight reflects the intensity of connection (resp. repulsion) between two nodes (with $w(e) = 0$ being a neutral relation).

- Sometimes (as in Chapter 3) we will only have a qualitative (rather than quantitative) weight, to reflect a strong or weak relation (tie).

- Analogous to shortest paths in an unweighted graph, we often wish to compute least cost paths, where the cost of a path is the sum of weights of edges in the path.

The six degrees of freedom phenomena

There are two basic ways for finding someone in a social network.

- We could ask all of our friends to tell all of their friends to tell all of their friends. . . (i.e. a traditional chain letter) that I am looking for person X.

- Now say assuming your online social network has a "broadcast to all" feature, this can be done easily but it has its drawbacks. Drawbacks?

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- Clearly if everyone cooperates, the broadcast method ensures the shortest path to the intended target X in the leveled tree/graph of reachable nodes.

Reachable nodes without triadic closure

- If there is no triadic closure (i.e. your friends are not mutual friends, etc.), it is easy to see why every path is a shortest path to everyone in the network.
- Consider the number of people that you could reach by a path of length at most t if every person has say at least 5 friends.

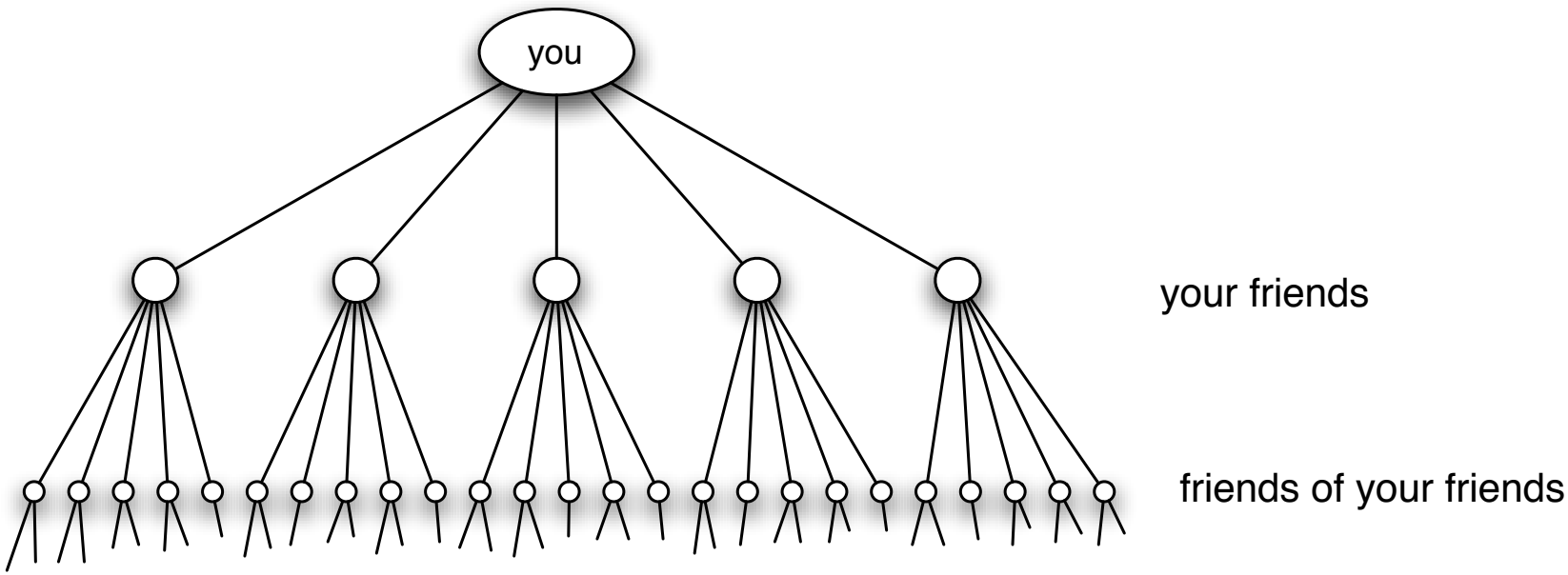


Figure: Pure exponential growth produces a small world [Fig 20.1 (a), E&K]

Reachable nodes with triadic closure

- Given that our friends tend to be mostly contained within a few small communities, the number of people reachable will be much smaller.

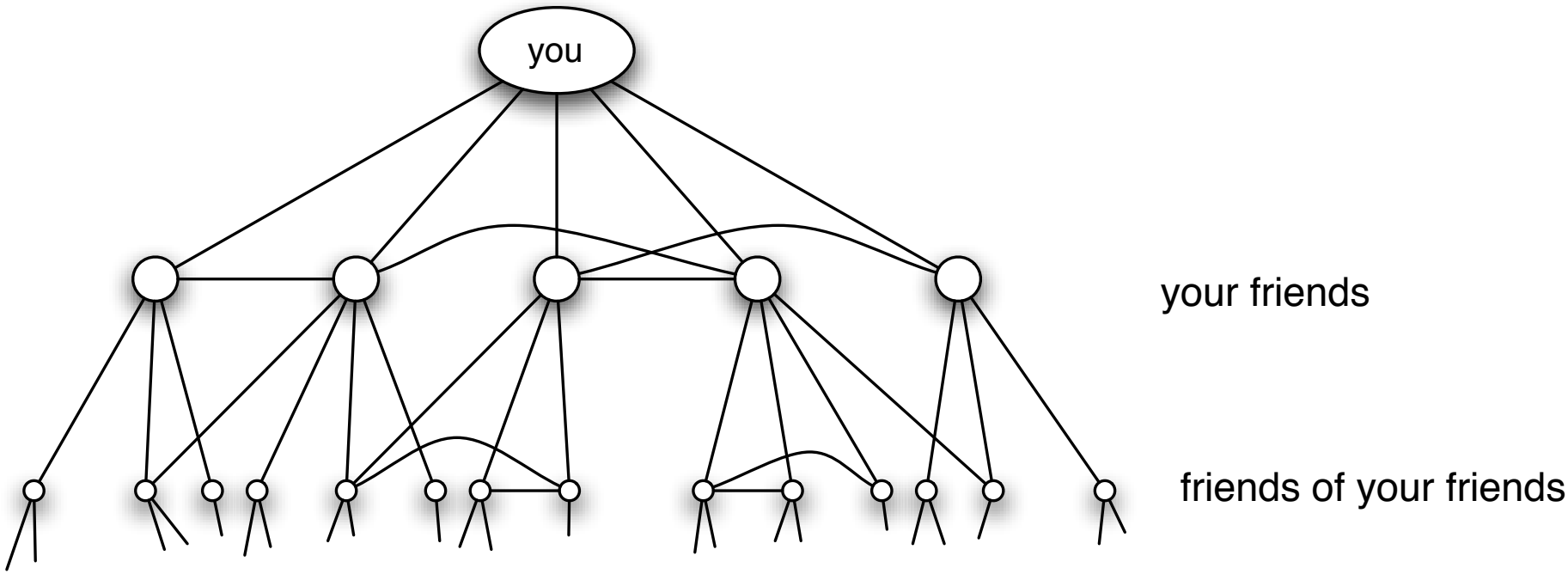


Figure: Triadic closure reduces the growth rate [Fig 20.1 (b), E&K]

The Watts-Strogatz model

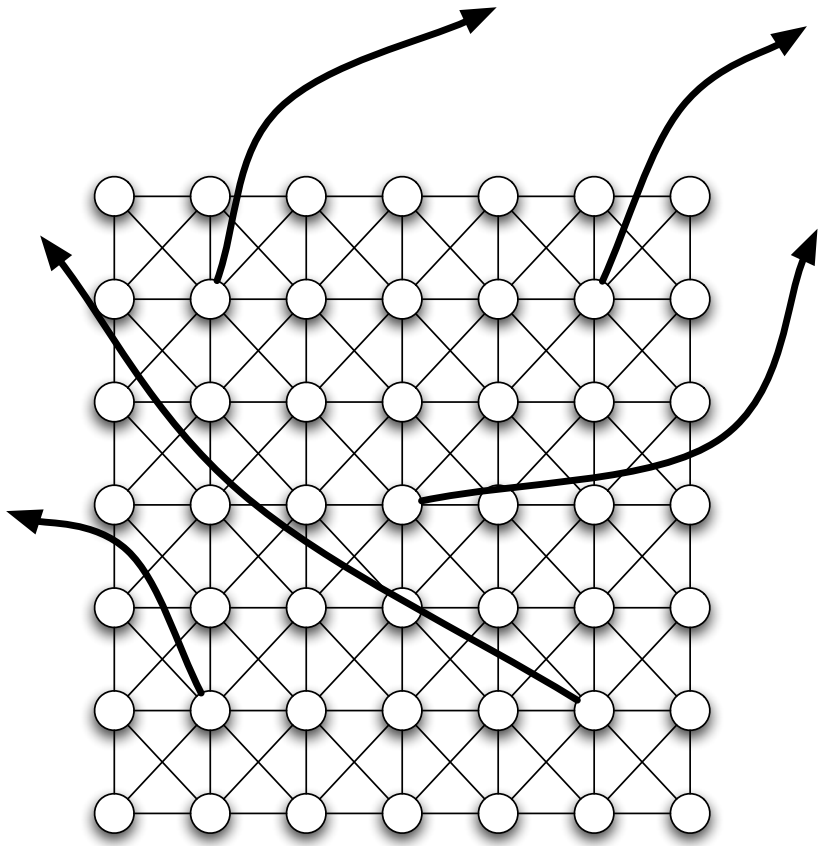
- Is it possible to have extensive triadic closure and still have short paths?

- Homophily is consistent with triadic closure especially for strong ties whereas weak ties can connect different communities and thereby provide the kind of branching that yields short paths to many nodes.

- One stylized model to demonstrate the effect of these different kinds of ties is the Watts-Strogatz model, which considers nodes lying in a two dimensional grid and then having two types of edges:
  - Short-range edges to all nodes within some small distance r. This captures an idealized sense of homophily
  - A small number of random longer-distance edges to other nodes in the network; in fact, one needs very few such random edges to achieve the effect of short paths.

Very few random edges are needed

- A k by k "town" with probability $1/k$ that a person has a random weak tie.
- This would be sufficient to establish short paths.



[Fig 20.3, E&K]

But how does this explain the ability to find people in a decentralized manner

- In the Watts-Strogatz type of model, we can use the random edges (in addition to the short grid edges) and the geometric location of nodes to keep trying to reduce the grid distance to a target node.
  - This is analogous to the Milgram experiment where individuals seem to use geographic information to guide the search.
  - However, completely random edges does no reflect real social networks

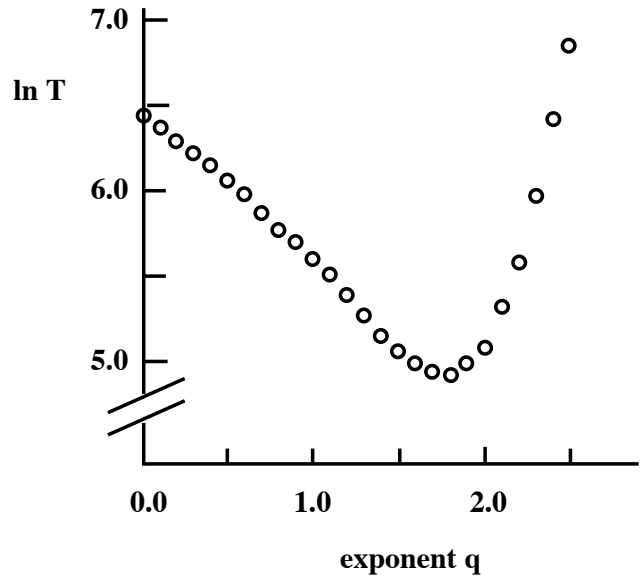
- Furthermore, having uniformly random edges will not work in general as:
  - Completely random edges (i.e. going to a random node anywhere in the network) are too random.
  - A random edge in an $n \times n$ grid is likely to have grid distance $\Theta(n)$.
  - Without some central guidance, such random edges will essentially just have us bounce around the network causing a substantially longer path to the target than the shortest path.

A modification of the model

- Random edges outside of ones "close community" are still more likely to reflect some relation to closeness.
- So assume as in the Watts-Strogatz model, from every node v we have edges to all nodes x within some grid distance r from v.
- And now in addition random edges are generated as follows: we (independently) create an edge from v to w with probability proportional to $d(v,w)^{-q}$ where $d(v,w)$ is the grid distance from v to w and $q \geq 0$ is called the clustering exponent.
- The smaller $q \geq 0$ is, the more completely random is the edge whereas large $q \geq 0$ leads to edges which are not sufficiently random and basically keeps edges within or very close to ones community.
- What is the best choice of $q \geq 0$?

So what is a good or the best choice of the clustering exponent q?

- It turns out that in this 2-dimensional grid model decentralized search works best when $q = 2$. (This is a result that holds and can be proven for the limiting behaviour, in the limit as the network size increases.)



[Fig 20.6, E&K]

- Simulation of decentralized search in the grid-based model with clustering exponent q.
- Each point is the average of 1000 runs on (a slight variant of) a grid with 400 million nodes.
- The delivery time is best in the vicinity of exponent $q = 2$, as expected.
- But even with this number of nodes, the delivery time is comparable over the range between 1.5 and 2.

More precise statements of Kleinberg's results on navigation in small worlds

The Milgram-like experiment

- Consider a grid network and construct (local contact) directed edges from each node u to all nodes v within grid distance $d(u, v) = k > 1$.
- Also probabilistically construct m (long distance) directed edges where each such edge is chosen with probability proportional to $d(v, w)^{-q}$ for $q \geq 0$.
- We think of k and m as constants and consider the impact of the clustering exponent q as the network size n increases.
- We assume that each node knows its location and the location of its adjacent edges and its distance to the location of a target node t.
- The Milgram-like experiment is that each node it *tries* (without knowing the entire network) to move from a node u to a node v that is closest to t (in grid distance).

Reflection on the Kleinberg-Milgram model

As we said at the start of this topic, the real surprise is that a "short" (but not shortest) path is (probably wrt to the randomly generated network) being found by a decentralized search.

It is true that each node will pursue a "greedy strategy" but this is different than say Dijkstra's least cost/distance algorithm which entails a centralized search.

Navigation in small worlds results

Theorem

(J. Kleinberg 2000)

(a) *For $0 \leq q < 2$, the (expected) delivery time T of any "decentralized algorithm" in the $n \times n$ grid-based model is $\Omega\left(n^{\frac{2-q}{3}}\right)$.*

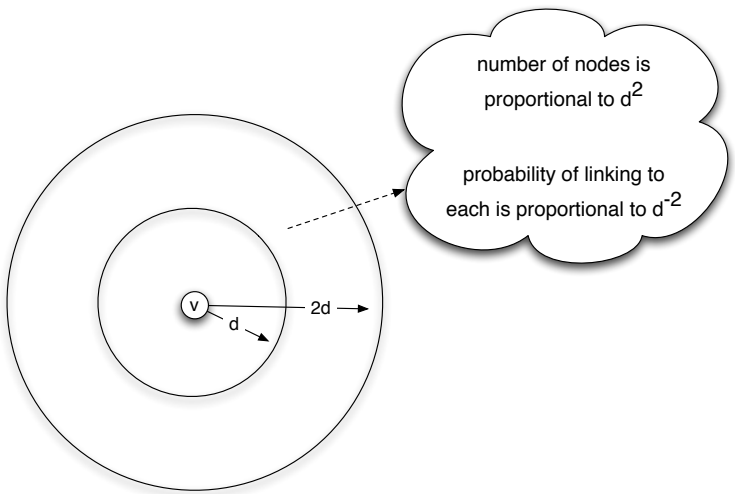
(b) *For $q = 2$, there is a decentralized algorithm with delivery time $O(\log n)$.*

(c) *For $q > 2$, the delivery time of any decentralized algorithm in the grid-based model is $\Omega\left(n^{\frac{q-2}{q-1}}\right)$.*

(The lower bounds in (a) and (c) hold even if each node has an arbitrary constant number of long-range contacts, rather than just one.)

Intuition as to why $q = 2$ is best for the grid

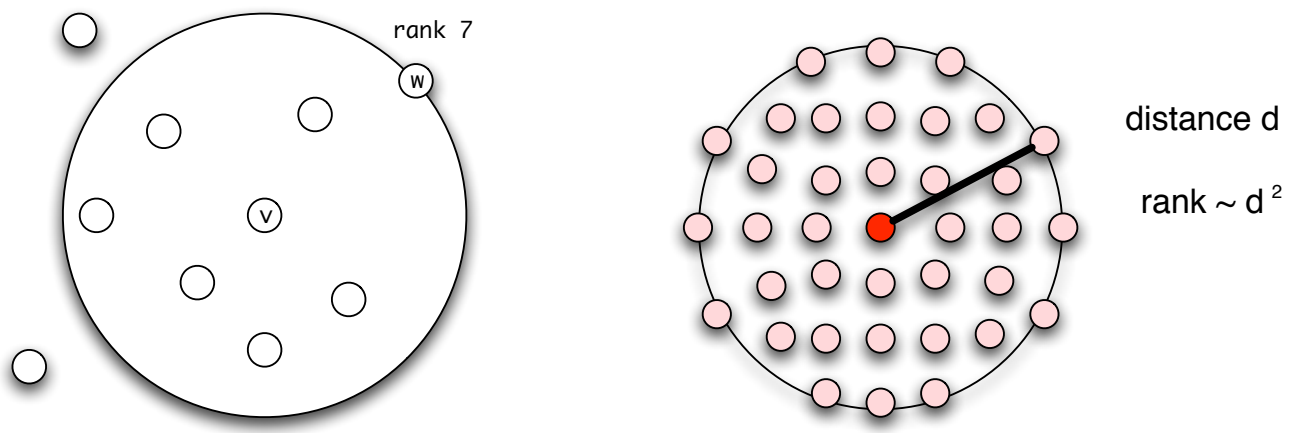
- It is instructive to see why this choice of q provides links at the different "scales of resolution" seen in the Milgram experiment.
- That is, if D is the maximum distance to be travelled, then we would like links with distances between d and $2d$ for all $d < \log D$
- Given that we have a 2-dimensional grid, the number of points with distance say d from a given node v will be $\approx d^2$.
- We are choosing such a node with probability proportional to $1/d^2$ and hence we expect to have a link to some node whose distance from v is between d and $2d$ for all d.



[Fig 20.7, E&K]

More realistic (nonuniformly spread) population data

- In the grid model, the population density is completely uniform which is not what one would expect in real data.
- How can this $1/d^2$ (inverse-square) distribution be modified to account for population densities that are very non-uniform?
- The idea is to replace distance $d(v, w)$ from v to w by the rank of w relative to v.
  - For a fixed v, define the $rank(w)$ to be the number of nodes closer to v than w.
  - In the 2D grid case, when $d(v, w) \sim d$, then $rank(w) \sim d^2$.

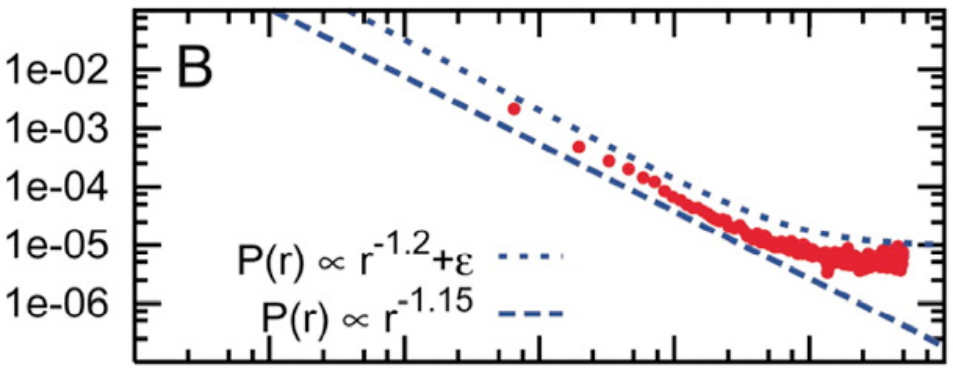


[Fig 20.9, E&K]

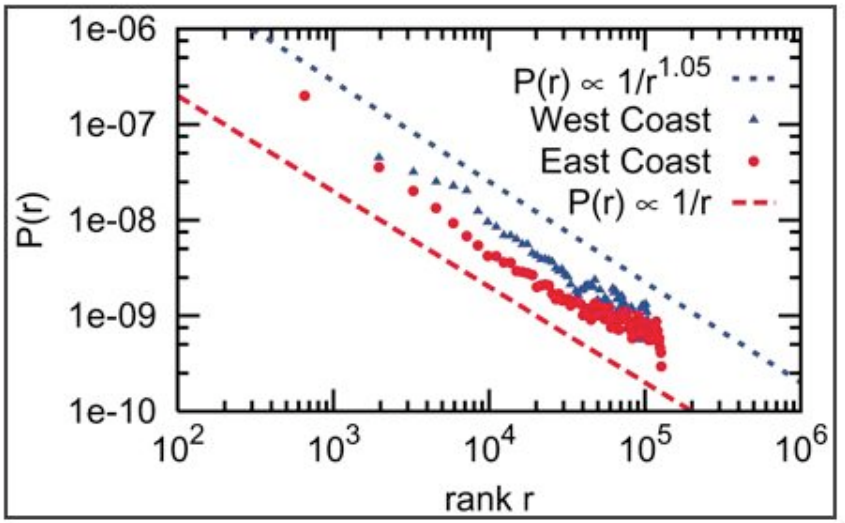
More realistic geographic data continued

- We can then restate the inverse-square distribution by saying that the probability that v links to w is proportional to $1/rank(w)$.
- Using zip code information, for every pair of nodes (500,000 users on the blogging site LiveJournal) one can assign ranks.
- Liben-Nowell et al did such a study in 2005, and then for different rank values examined the fraction f of edges that are actually friends.
- The theory tells us that this fraction f should be a decreasing function proportional to $1/rank$.
- That is, $f \sim rank^{-1}$. Taking logarithms, $\log f \sim (-1) \log rank$.

More realistic (LiveJournal) friendship data



(a) *Rank-based friendship on LiveJournal*



(b) *Rank-based friendship: East and West coasts*

[Fig 20.10, E&K]

- In Figure 20.10 (a), the Lower (upper) line is exponent $= -1.15$ (resp. -1.12).
- In Figure 20.10 (b), the Lower (upper) line is exponent $= -1.05$ (resp. -1). The red data is East Coast data and the blue data is West Coast data.

Liben-Nowell: practice closely matches theory

Liben-Nowell prove that for "essentially" any population density (i.e. no matter where people are located) if links are randomly constructed so that the probability of a friendship is proportional to $rank^{-1}$, then the resulting network is one that can be efficiently searched in a decentralized manner.

That is, Kleinberg's result for the grid generalizes. This is a rather exceptional result in that the abstraction from d^{-2} to $rank^{-1}$ is not at all an obvious generalization.

How surprised should we be that natural populations locate themselves in this probabilistic manner since there is no centralized organizing mechanism that is causing this phenomena?

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The EK text refers to a 2008 article by Oscar Sandberg who analyzes a network model where decentralized search takes place which in turn causes links to "re-wire" so as to fascilitate more efficient decentralized search.

It remains an intringing question as to the extent this does happen in social networks and the implicit mechanisms that would cause networks to evolve this way.

IP addresses and the TCP/IP routing protocol

For those taking (or having taken) a computer networks course, you can observe how IP addresses allow the IP transmission protocol to send messages along a decnetralized route.

TCP/IP originated in the earlyh 1980's which is much after Milgram but well before Strogatz and Kleinberg. To what extent was the TCP/IP protocol and IP addresses motivated by Milgram's work?

But perhaps postal codes are the original motivation?

Aside Interesting ideas usually have a history and the best we can do is document some of the major events in the adoption of any important idea.

The Backstrom et al rank-based study

- Backstrom et al study US Facebook 2010 geographic user data.
  1. Roughly 100 million users
  2. About 6% of which enter home address info and of that population about 60% can be parsed into longitude and lattiude information.
  3. This gave a set of 3.5 million users (of which 2.9 million had at least one friend with a well specified address and each of these 2.9 million users had an average of 10 friends with specified addresses resulting in 30.6 million edges.
  4. Although a small part of Facebook, this 2.9 million person "geolocated data set" is sufficently large and representative for experimental study.

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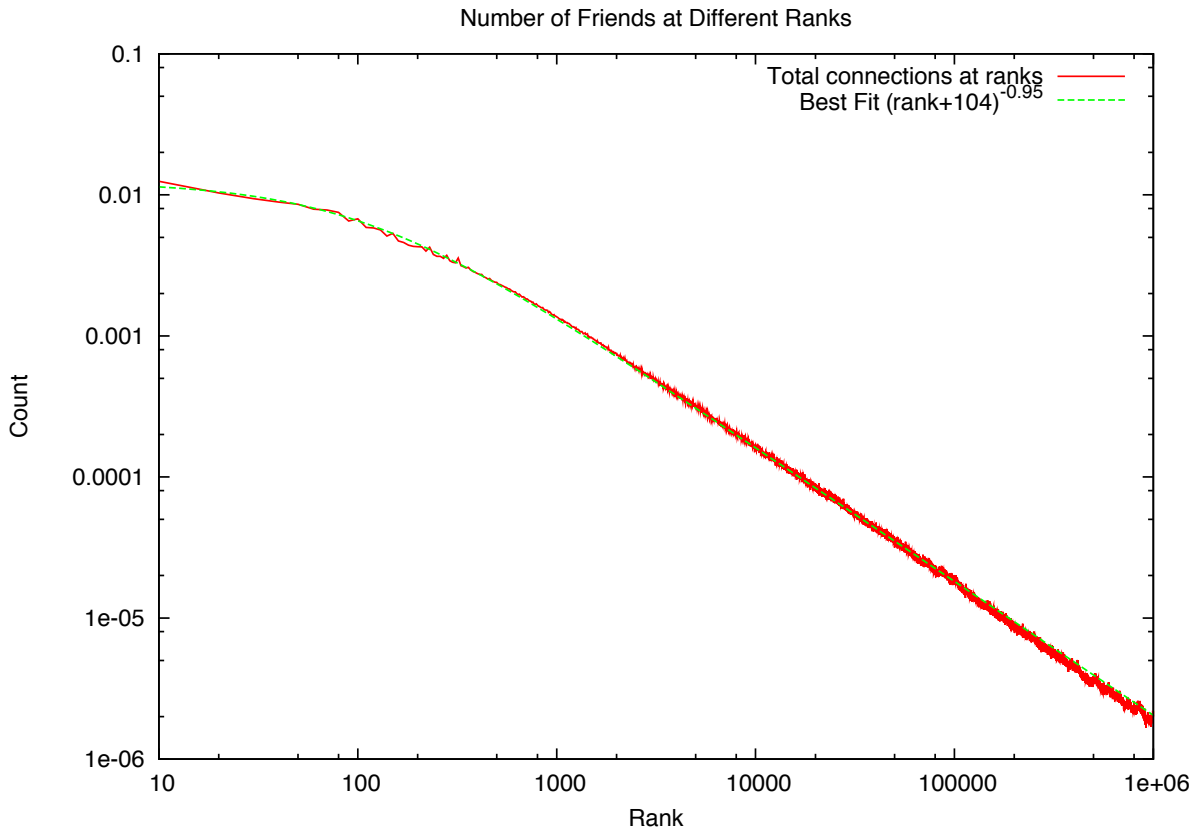
- They study probability of friendships vs distance and rank and how those probabilities depend on population densities for where people live. This study provides more evidence as to the power law relation between distance/rank and probability ($\approx rank^{-.95}$) of friendship.
- Furthermore, they utilize this relationship between friends and distance to create an algorithm that will predict the location of an individual from a small set of users with known locations. They claim their algorithm can predict geographic locations better than using IP information!

Number of friends wrt. rank



[Figure 9 from Backstrom et al]

Predicting locations

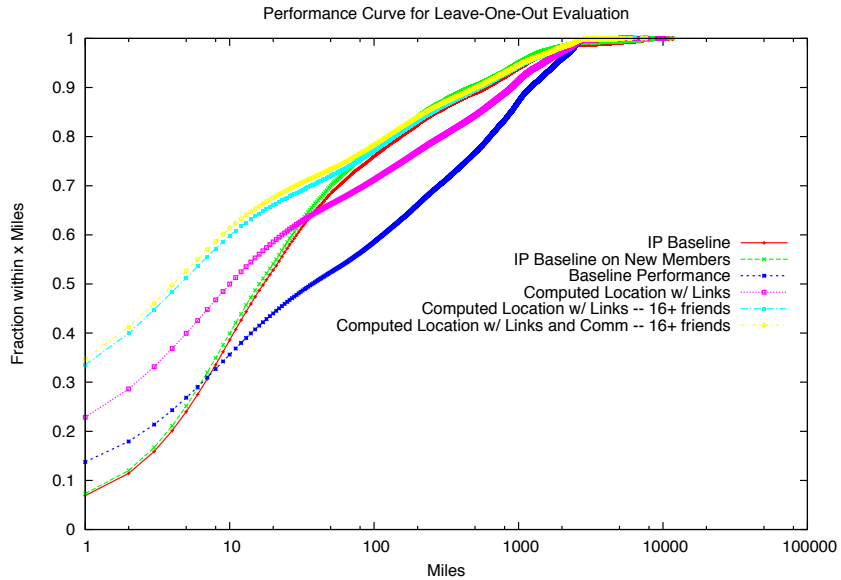


Figure 11: Location Prediction Performance. This figure compares external predictions from an IP geolocation service, the same service constrained to users who have recently updated their address, a baseline of randomly choosing the location of a friend, along with three predictions: our algorithm with all links, for users with 16+ friends, and finally for users with 16+ friends constraining to only those with whom they have communicated recently.

[Figure 11 from Backstrom et al]

From geographic distance to social distance

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- But, of course, belonging to a small group of people in a course, is different than attending the same University, and speaking Mandarin is different than being interested in Esperanto.
- So the suggestion is made that we define social distance $s(v, w)$ between individuals v, w to be the minimum size of a common focus.

Smallest size shared focus as a distance measure

- Kleinberg (2001) gives theoretical results indicating that when friendships follow a distribution proportional to $1/s(v, w)$ then the resulting social network will support efficient decentralized search.

- This is somewhat verified in a study (by Adamic and Adar) of 'who talks to whom' friendship data (based on frequency of email exchanges) amongst a small group of HP employees.

- The focal groups are defined by the organizational hierarchy of the company.

- The Adamic and Adar 2005 study shows that the distribution for this friendship relationship is proportional to the inverse of $s(v, w)^{-3/4}$ so that it doesn't match as closely with the previous geographical rank based results but still observes a power law relation governing how social ties decrease with "distance".

Probability of email exchanges vs distance in the organizational hierarchy

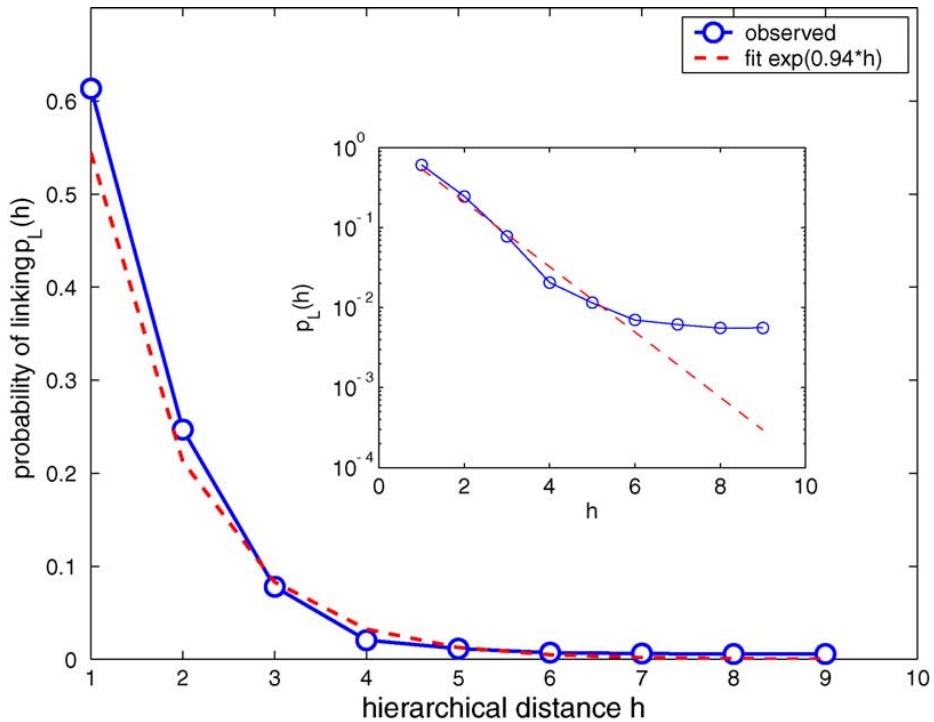

Fig. 4. Probability of linking as a function of the separation in the organizational hierarchy. The exponential parameter $\alpha = 0.94$, is in the searchable range of the Watts model (Watts et al., 2002).

[Figure 4 from Adamic and Adar]

Probability of email exchanges vs size of smallest common organizational unit

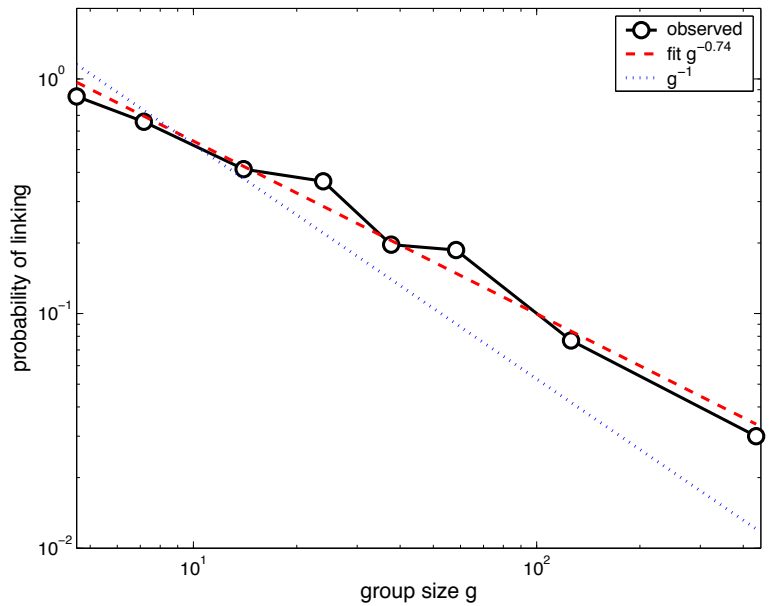

Figure 5: Probability of two individuals corresponding by email as a function of the size of the smallest organizational unit they both belong to. The optimum relationship derived in [7]is $p \sim g^{-1}$, g being the group size. The observed relationship is $p \sim g^{-3/4}$.

[Figure 5 from Adamic and Adar]

Final observations in chapter 20 of EK text

- The EK text suggests viewing the Milgram experiment as an example of decentralized problem solving (in this case solving a shortest path problem). An advertisement for distributed systems course.

- The EK text asks what other problem solving tasks might be amenable to such decentralized problem solving and how to analyze what can be done especially in large online networks.

- Finally the EK text briefly suggests the role of social status in determining the effectiveness of reaching a given target.
  - An email forwarding Milgram type 2003 study by Dodds et al shows that completion rates to all targets were low but were highest for "high status" targets and particularly small for "low status" targets.

- In section 12.6, the EK text speculates on structural reasons for the impact of status. This discussion leaves me with the sense that we are far from having any comprehensive understanding of such phenomena.