Great Ideas in Fair Division

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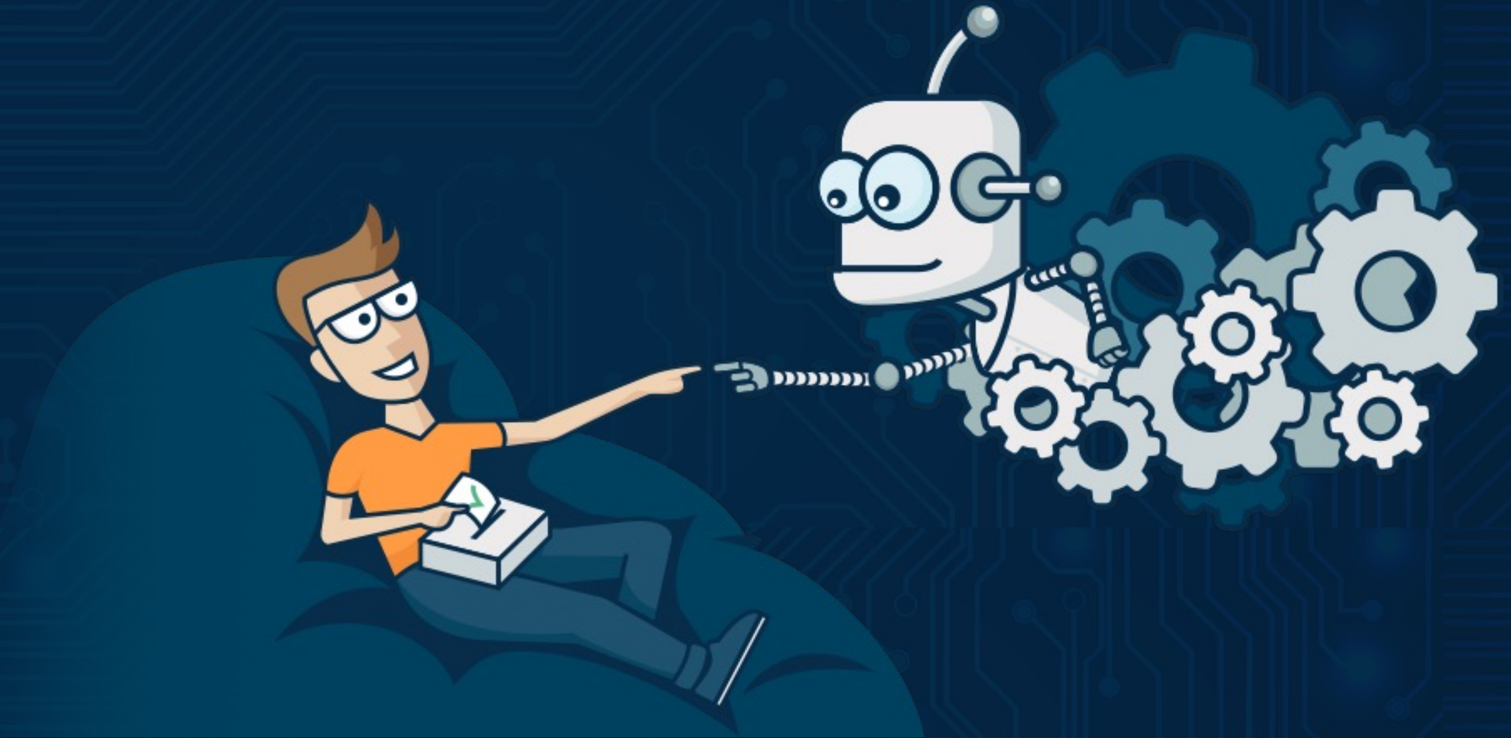
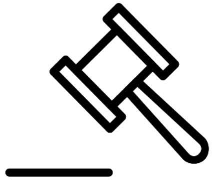


ALGORITHMS MAKING DECISIONS

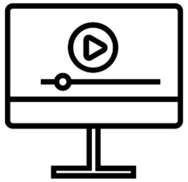
Loans

Bails



Self-Driving Cars

Ads



Hiring

Organ Exchange

Computational Social Choice

Algorithms for aggregating individual preferences
towards collective decisions

Reasonable Collective Decisions

Cake Cutting

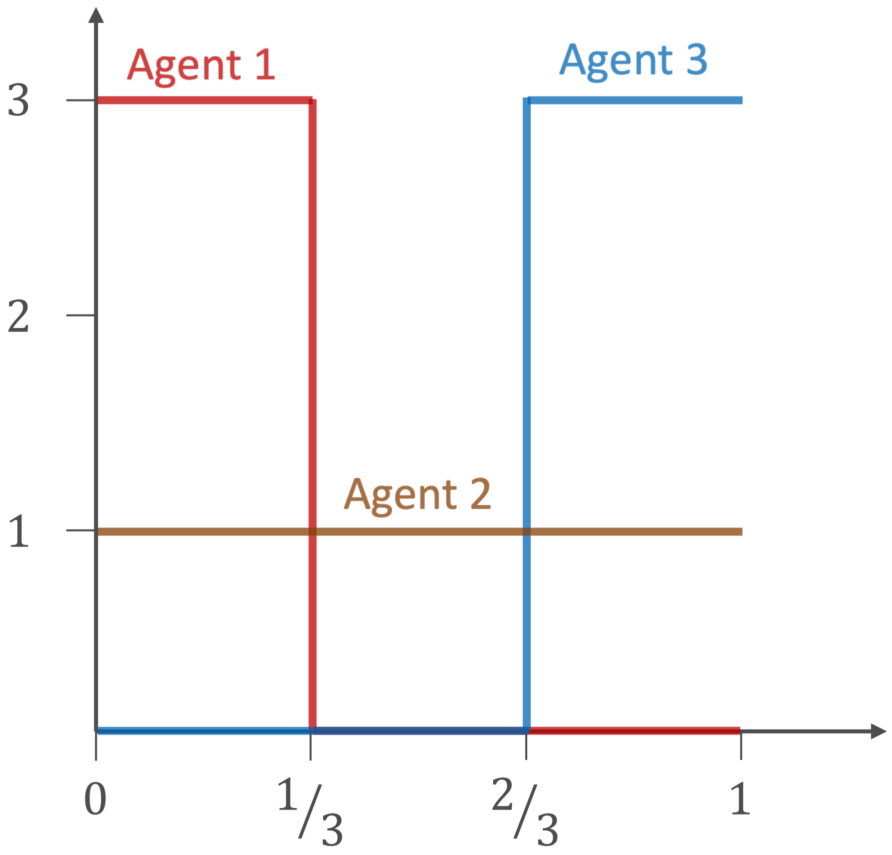
- Formally introduced by Steinhaus [1948]
- n people ("agents")
- Cake modeled as $[0,1]$
- Allocate the cake
    - $A_i \subseteq [0,1]$ given to agent i
        - E.g., $A_i = [0.1,0.3] \cup [0.5,0.9]$ is allowed
    - $A_i \cap A_j = \emptyset$ for all i, j

AGENT VALUATIONS

- Each agent i has an integrable density function $f_i: [0,1] \to \mathbb{R}_+$
- $v_i(X) = \int_{x \in X} f_i(x) dx$
- Normalization: $\int_0^1 f_i(x) dx = 1$
  - Without loss of generality

EXAMPLE

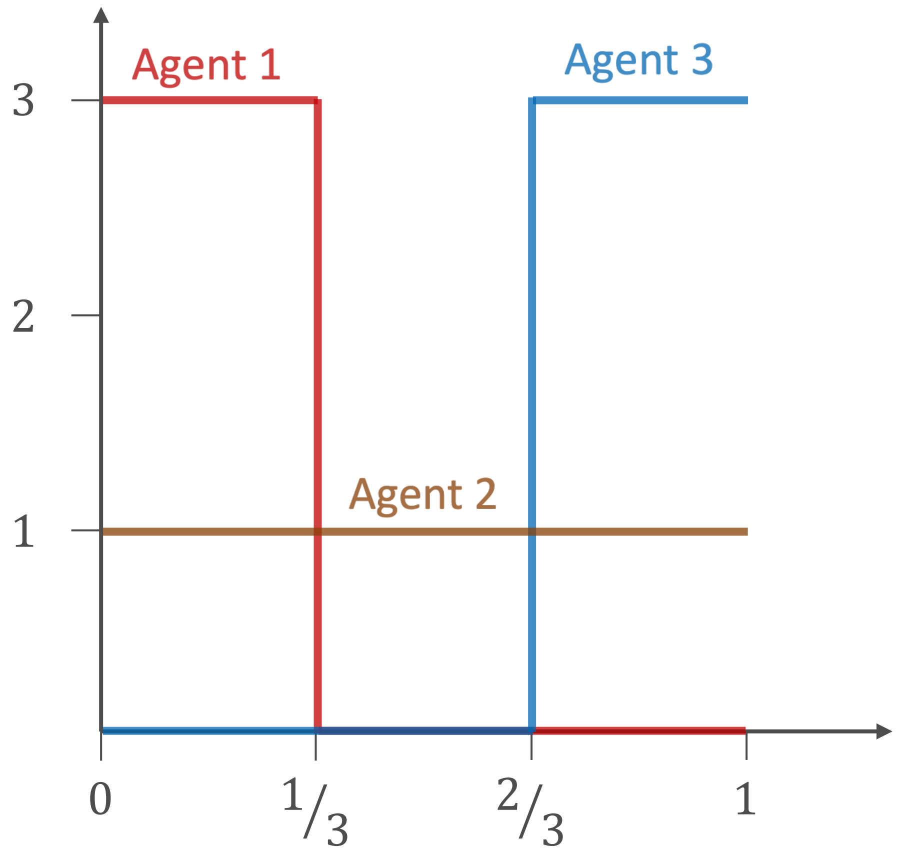
- Value density functions



- Agent 1 wants $[0, {}^{1}/_{3}]$ uniformly and does not want anything else
- Agent 2 wants the entire cake uniformly
- Agent 3 wants $[{}^{2}/_{3}, 1]$ uniformly and does not want anything else

EXAMPLE

- Value density functions

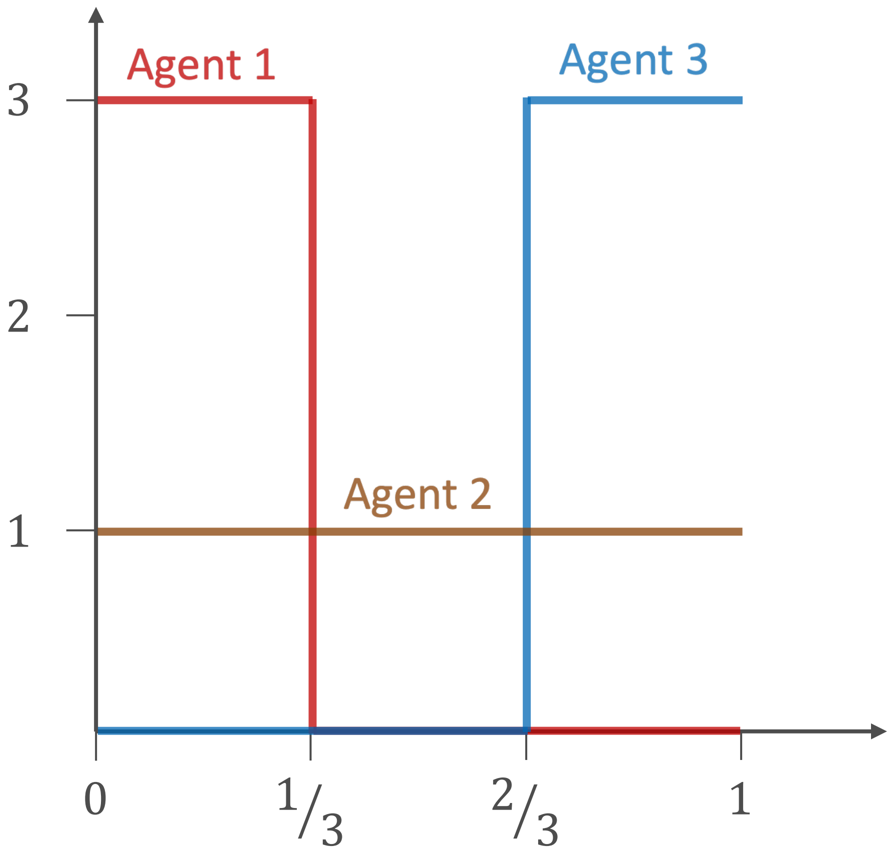


- Consider the following allocation
  - $A_1 = [0, {}^1/_9] \Rightarrow v_1(A_1) = {}^1/_3$
  - $A_2 = [{}^1/_9, {}^8/_9] \Rightarrow v_2(A_2) = {}^7/_9$
  - $A_3 = [{}^8/_9, 1] \Rightarrow v_3(A_3) = {}^1/_3$

- Each of three agents is getting at least one-third of their value, which seems fair in some sense
- But agent 1 and 3 are envious of agent 2, and would want to get his allocation instead

Example

- Value density functions



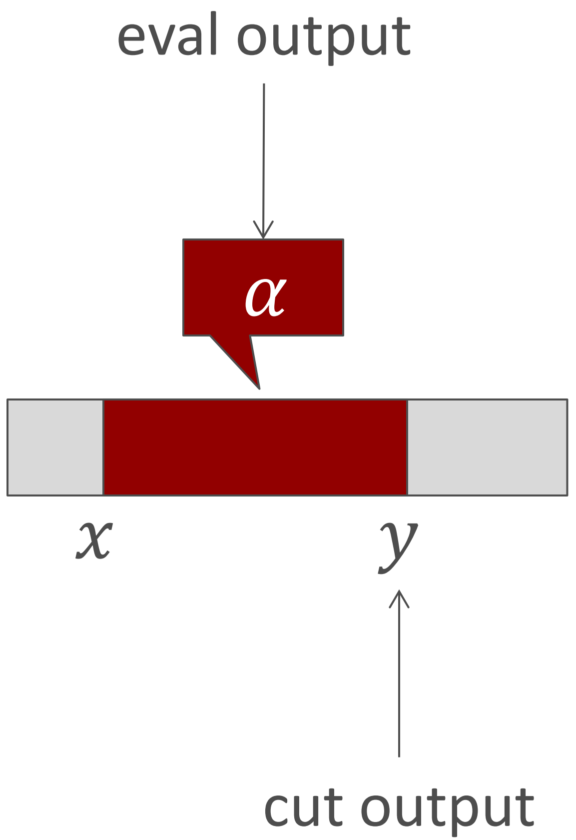
- Consider the following allocation
  - $A_1 = [0, {}^1/_6] \Rightarrow v_1(A_1) = {}^1/_2$
  - $A_2 = [{}^1/_6, {}^5/_6] \Rightarrow v_2(A_2) = {}^2/_3$
  - $A_3 = [{}^5/_6, 1] \Rightarrow v_3(A_3) = {}^1/_2$

- Now agent 1 and 3 are not envious of what agent 2 is given, even though agent 2 has more utility than them

COMPLEXITY

- Inputs are functions
  - Infinitely many bits may be needed to fully represent the input
  - Query complexity is more useful

- Robertson-Webb Model
  - $\text{Eval}_i(x, y)$ returns $v_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns y such that $v_i([x, y]) = \alpha$



Three Classic Fairness Desiderata

- Proportionality (Prop): $\forall i \in N: v_i(A_i) \geq 1/n$
  - Each agent should receive her "fair share" of the utility.
- Envy-Freeness (EF): $\forall i, j \in N: v_i(A_i) \geq v_i(A_j)$
  - No agent should wish to swap her allocation with another agent.
- Envy-freeness implies proportionality (Why?)

Proportionality

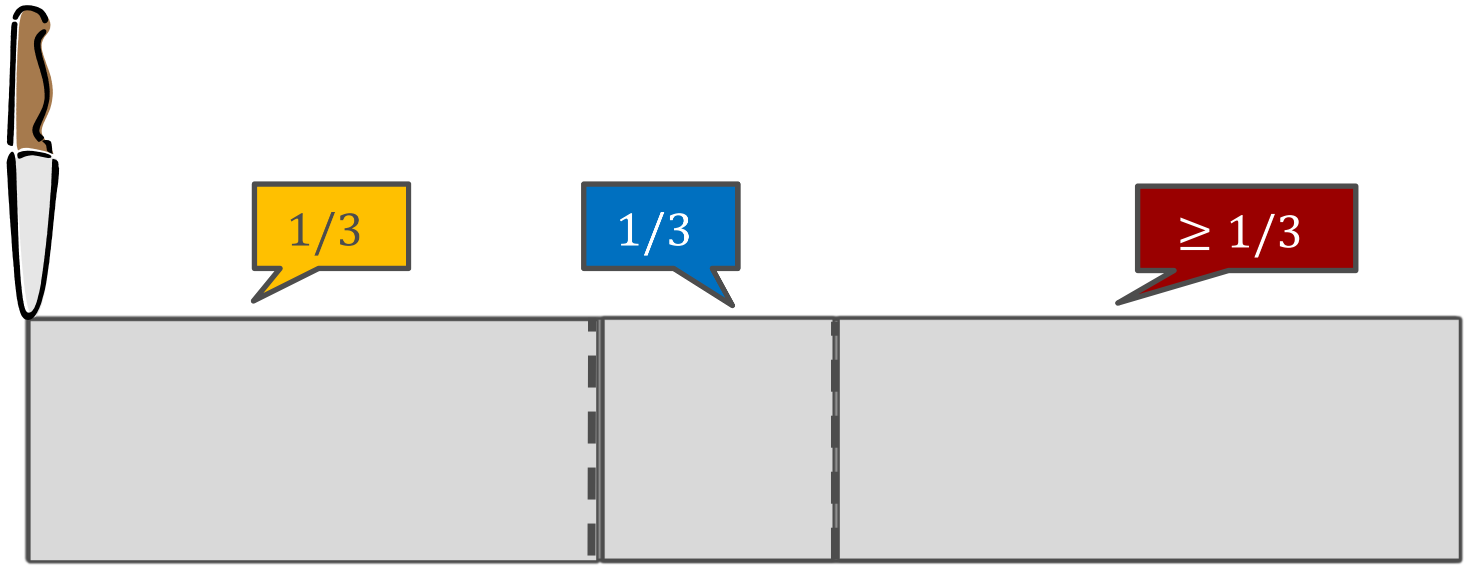
PROPORTIONALITY : $n = 2$ AGENTS

- CUT-AND-CHOOSE
  - Agent 1 cuts the cake at x such that $v_1([0, x]) = v_1([x, 1]) = 1/2$
  - Agent 2 chooses the piece that she prefers.

- Elegant protocol
  - Envy-free for 2 agents
  - Needs only one cut and one eval query (optimal)

- More agents?

Proportionality: Dubins-Spanier



Animation Credit: Ariel Procaccia

Proportionality: Dubins-Spanier

- Dubins-Spanier
  - Referee starts a knife at 0 and moves the knife to the right.
  - Repeat: When the piece to the left of the knife is worth $1/n$ to an agent, the agent shouts "stop", receives the piece, and exits.
  - When only one agent remains, she gets the remaining piece.

- Can be implemented easily in Robertson-Webb model
  - When $[x, 1]$ is left, ask each remaining agent i to cut at y_i so that $v_i([x, y_i]) = 1/n$, and give agent $i^* \in \arg\min_i y_i$ the piece $[x, y_{i^*}]$

- Question: What is the asymptotic query complexity as a function of the number of agents n?

Complexity of Proportionality

- Theorem [Evan and Paz, 1984]:
  - There is a protocol that returns a proportional allocation in $O(n \log n)$ queries in the Robertson-Webb model.

- Theorem [Edmonds and Pruhs, 2006]:
  - Any protocol returning a proportional allocation needs $\Omega(n \log n)$ queries in the Robertson-Webb model.

Envy-Freeness

Envy-Freeness : Few Agents

- $n = 2$ agents : Cut-and-choose (2 queries)
- $n = 3$ agents : Selfridge-Conway (14 queries)

Gets complex pretty quickly!

Suppose we have three players **P1**, **P2** and **P3**. Where the procedure gives a criterion for a decision it means that criterion gives an optimum choice for the player.

1. **P1** divides the cake into three pieces he considers of equal size.
2. Let's call **A** the largest piece according to **P2**.
3. **P2** cuts off a bit of **A** to make it the same size as the second largest. Now **A** is divided into: the trimmed piece **A1** and the trimmings **A2**. Leave the trimmings **A2** to the side for now.
   - If **P2** thinks that the two largest parts are equal (such that no trimming is needed), then each player chooses a part in this order: **P3**, **P2** and finally **P1**.
4. **P3** chooses a piece among **A1** and the two other pieces.
5. **P2** chooses a piece with the limitation that if **P3** didn't choose **A1**, **P2** must choose it.
6. **P1** chooses the last piece leaving just the trimmings **A2** to be divided.

It remains to divide the trimmings **A2**. The trimmed piece **A1** has been chosen by either **P2** or **P3**; let's call the player who chose it **PA** and the other player **PB**.

1. **PB** cuts **A2** into three equal pieces.
2. **PA** chooses a piece of **A2** - we name it **A21**.
3. **P1** chooses a piece of **A2** - we name it **A22**.
4. **PB** chooses the last remaining piece of **A2** - we name it **A23**.

Envy-Freeness : Few Agents

- [Brams and Taylor, 1995]
  - The first finite (but unbounded) protocol for any number of agents
- [Aziz and Mackenzie, 2016a]
  - The first bounded protocol for 4 agents (at most 203 queries)
- [Amanatidis et al., 2018]
  - A simplified version of the above protocol for 4 agents (at most 171 queries)

ENVY-FREENESS

- Theorem [Aziz and Mackenzie, 2016b]
  - There exists a bounded protocol for computing an envy-free allocation with n agents, which requires $O(n^{n^{n^{n^{n^n}}}})$ queries

- Theorem [Procaccia, 2009]
  Any protocol for finding an envy-free allocation requires $\Omega(n^2)$ queries.

Open Problem

Bridge the gap between $O(n^{n^{n^{n^{n^n}}}})$ upper bound and $\Omega(n^2)$ lower bound for envy-free cake-cutting

INDIVISIBLE GOODS

- Estate (inheritance) division
- Divorce settlement
- Friends splitting jointly purchased items
- …

spliddit

DIVIDE: RENT FARE CREDIT GOODS TASKS | ABOUT FEEDBACK

PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

Share Rent

Split Fare

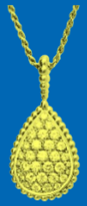
Assign Credit

Divide Goods

Distribute Tasks

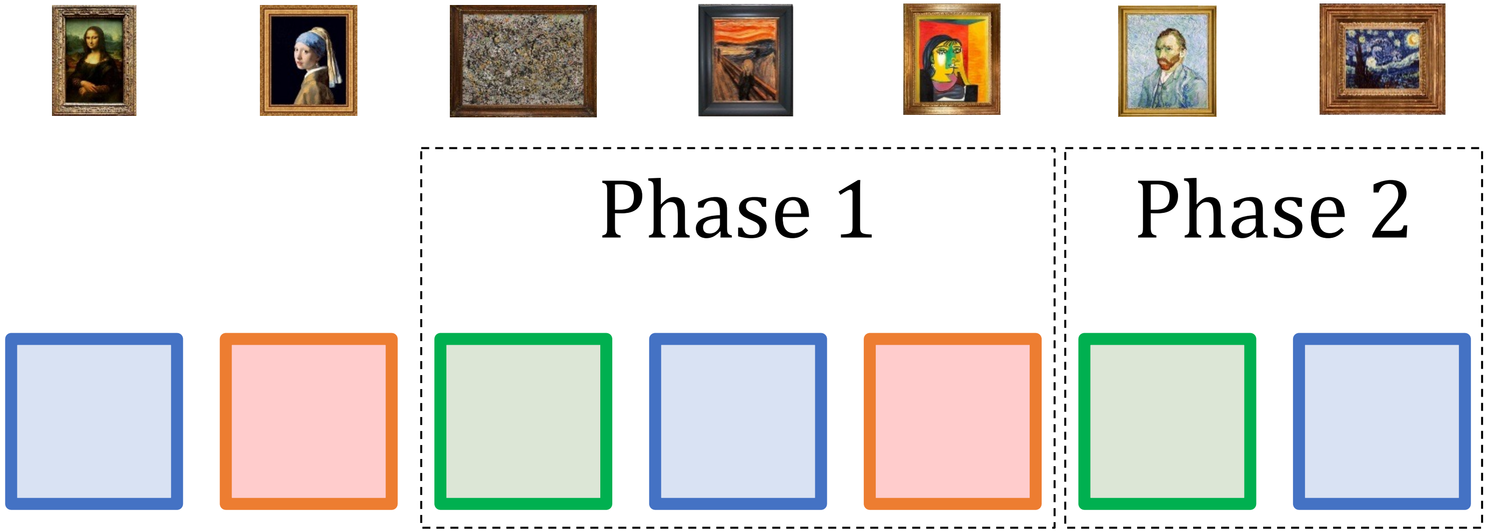
Suggest an App

Setting

				
	67	150	256	12
	150	27	39	53
	25	121	352	5

Approximate Envy-Freeness

- Envy-Freeness Up To One Good (EF1)
  - No agent envies another agent if we ignore at most one good allocated to the envied agent
  - $\forall i,j \in N \ \exists^* g \in A_j : v_i(A_i) \geq v_i\left(A_j \setminus \{g\}\right)$
- Simple round robin achieves this:



EFFICIENCY

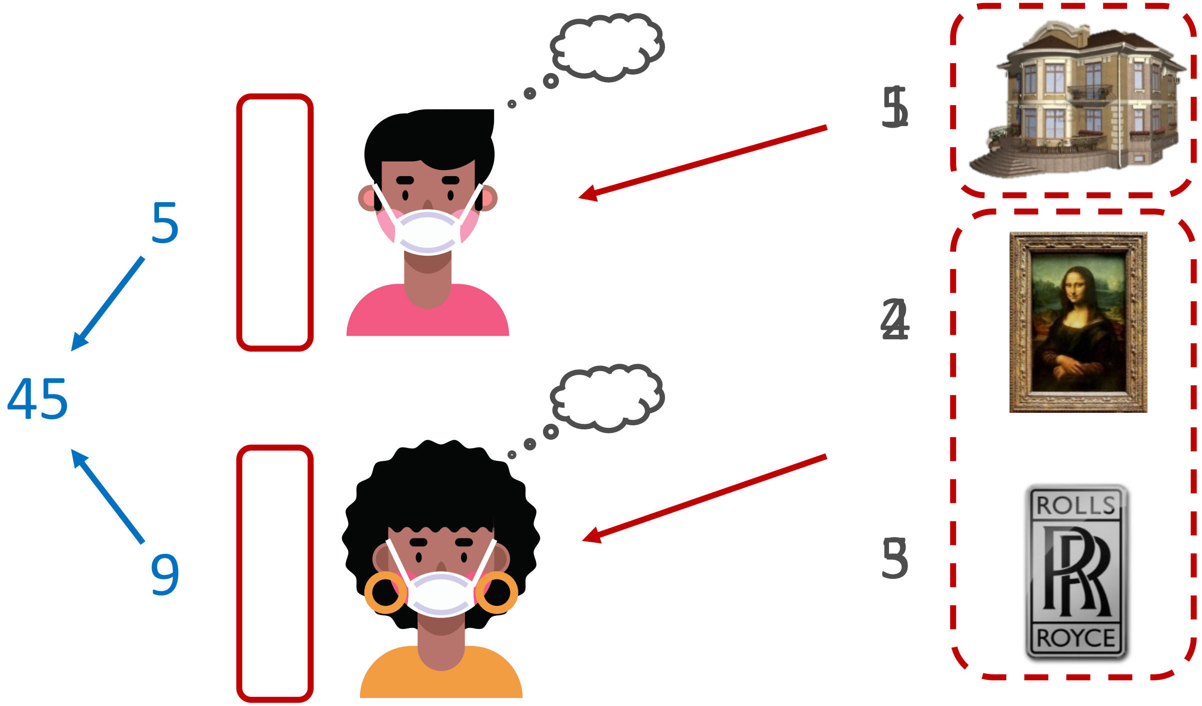
- Pareto optimality (PO)
  - No other allocation should give more utility to every agent
  - $\nexists B \left(\forall i : v_i(B_i) > v_i(A_i)\right)$

- Round robin violates PO!

- Does there always exist an allocation that is both fair (EF1) and efficient (PO)?

Maximum Nash Welfare

- **Idea:** Maximize the Nash welfare $\prod_i v_i(A_i)$



Maximum Nash Welfare

Theorem [Caragiannis, Kurokawa, Procaccia, Moulin, S, Wang, 2016]

Maximizing Nash welfare satisfies EF1 and PO.

Open Questions

- Computation
  - Open Question: Can we compute an EF1+PO allocation in polynomial time?
    - Possible in pseudo-polynomial time [Barman et al., 2018]

- Envy-freeness up to any good (EFX)
  - No agent envies another agent if we ignore any good allocated to the envied agent
  - $\forall i, j \in N \;\; \forall g \in A_j : v_i(A_i) \geq v_i\left(A_j \setminus \{g\}\right)$
  - Open Question: Does there always exist an EFX allocation?
    - It exists for three agents [Chaudhury et al., 2020]

THANK YOU