Great Ideas in Computing

University of Toronto CSC196
Fall 2021

Week 10: November 21-25 (2022)

Announcements and todays agenda

Announcements

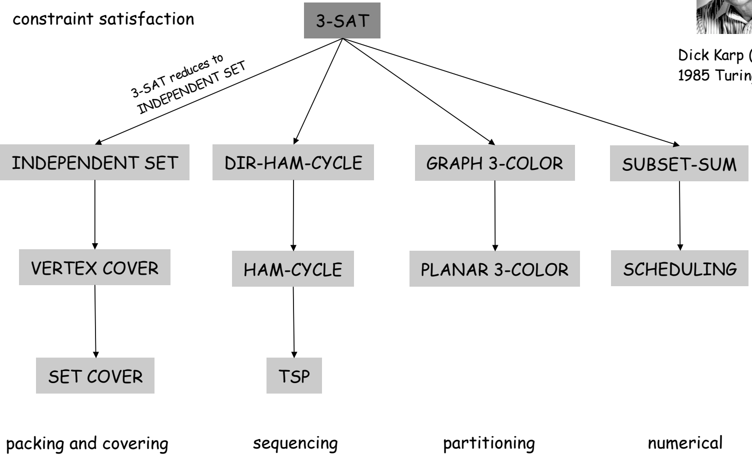
- The next and final quiz is scheduled for this Friday, November 25.
- I am changing the due date for Assignment 4 to be Monday, December 5.
- I recommend watching a PBS/NOVA presentation on the great mathematical idea of the number 0 and ∞. https://www.pbs.org/wgbh/nova/video/zero-to-infinity/

This weeks agenda

1. We will finish up the immediate discussion of the $P \neq NP$ conjecture/hypothesis. In particular, we will discuss:
    - An example in the Karp tree of a not so obvious transformation: 3-SAT $\leq_T^{poly}$ Independent Set. (In Cook's paper, it was stated as 3-SAT $\leq_T^{poly}$ Clique.) I will omit the *poly* superscript with the understanding that we are talking about polynomial time reductions and transformations.
    - Reducing Search and Optimization problems to a corresponding decision problem.
    - The $NP \neq co-NP$ conjecture
    - Can randomization help?
2. We begin complexity based cryptography.

A tree of reductions/transformations

Polynomial-Time Reductions



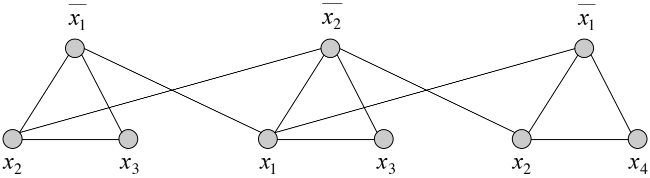
Dick Karp (1972)
1985 Turing Award

3SAT reduces to Independent Set

Claim

3SAT $\leq_\tau$ Independent Set

- Given an instance F of 3SAT with k clauses, we construct an instance (G, k) of Independent Set that has an independent set of size k iff F is satisfiable.
- G contains 3 vertices for each clause; i.e. one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Optimization problems

Each of these NP problems has an associated search or optimization problem. For example, the *Vertex-Cover* problem is usually expressed as the following optimization problem:
Given a graph $G = (V, E)$, find a minimum size vertex cover for G; that is, a subset $V' \subset V$ such that for every edge $e = (u, v) \in E$, either $u \in V'$ or $v \in V'$. This is the inculusive "or" so that it is possible that both u, v are in V'.

If we can solve the optimization problem efficiently, we can immediately solve the decision problemr?. Does everyone understand this?

What is not as immediate, is the fact that if we can solve the *Vertex-Cover* decision problem then we can solve the *Vertex-Cover* optimization problem.

We would do this by first determining (using the decision problem) the size of the minimum vertex cover. Does everyone see how to do this?

The vertex-cover optimization problem continued

Suppose k is the size of the minimum vertex cover. We want to create a vertex cover V' one vertex at a time starting with $V' = \varnothing$. We iteratively decide for each vertex v, whether or not we can include $v \in V'$. That is, we determine if we can remove v and all its adjacent edges and if the resulting graph $\tilde{G}$ has a vertex cover of size $k - 1$ then we add v to the cover V' that we are creating. We then continue with the graph $\tilde{G}$ trying to create a cover if size $k - 1$. If $\tilde{G}$ does not have a vertex cover of size $k - 1$, we go back to graph G and try another node u to see if $\tilde{G}$ has a vertex cover of size $k - 1$ if we remove u and its adjecent edges. We keep searching for a vertex x that we can remove from G and add to V'.

Note that while we can "probably" restrict attention to $\leq_{Karp}$ polynomial time transformations for the purpose of showing new problems are NP-complete, we are using the more general $\leq_{Cook}$ polynomial time reductions to reduce the optimization problem to the decision problem.

Another conjecture: $NP \neq$ co-NP

FACT: If L is NP-complete wrt $\leq_{Karp}$ then $\bar{L} \in NP$ if and only if $NP =$ co-NP

There is another widely believed conjecture again based on the inability of experts to show that $\bar{L} \in NP$ for any NP-complete problem which states that $NP \neq$ co-NP. For example, as stated before, we do not believe there is a "short" certificate for showing that a graph does *not* have a Hamiltonian cycle.

As I mentioned before, we believe factoring intergers is not polynomial time computable. In fact, there is a sense in which we believe it is not polynomial time computable "on average" (whereas the basic theory of NP completeness is founded on worst case analysis).

Surprisingly, co-$FACTOR$ is in NP. That is, given an input (N, k), we can provide a certificate verifying that N does *not* have a proper factor $m \leq k$.

Since co-$FACTOR$ is in NP, and we conjecture that $NP \neq$ co-NP, this leads us then to believe that $FACTOR$ is in $NP \setminus P$ but *not* NP-complete.

Returning to the two different reductions

As far as I know, there is no proof that the two reductions are different but there is good reason to believe that they are different in general.

- Clearly $\bar{A} \leq_{Cook} A$ for any language A.
- $A \leq_{Karp} B$ and $B \in NP$ implies $A \in NP$.
- Hence our assumption that $NP \neq co-NP$ implies that we *cannot* have $\bar{A} \leq_{Karp} A$ for any NP-complete A.

On the other hand as far as I know all known NP complete problems can be shown to be complete using transformations $\leq_{Karp}$.

I know of no compelling evidence that general reductions and transformations are different when resticted to the class NP.

NOTE: The general reduction concept is needed when reducing say a search or optimization problem to a decision problem (and indeed this is what we described for *Vertex-Cover* and we will be doing next for *SAT*). On the other hand, transformations are what we use for decision problems (i.e., languages).

Finding a certificate for an *NP*-complete problem

One might wonder if we can always efficiently *find* a certificate if we can decide whether or not a certificate exists. In fact, for *NP*-complete problems we can (Cook) reduce finding a certificate to deciding if a certificate exists.

Fact Let L be a *NP*-complete problem. We can prove that for every YES input instance x (where we know that a certificate exists wrt some verification predicate) that a certificate can be computed in polynomial time assuming we can solve the decision problem in polynomial time.

Of course, we do not believe that an *NP*-complerte decision problem can be solved in polynomial time so this is just a claim that it is sufficient to just focus on the decision problem.

As an another example, consider *SAT* and suppose F is satisfiable. That means we can set each propositional variable (to TRUE or FALSE) so that the formula evaluates to TRUE. So how do we find a satsifying truth assignment for F?

Finding a satisfying assignment for a formula F assuming $P = NP$

Once we assume $P = NP$, we would know that the decision problem for SAT is satisfiable. So we would first test if the given formula F is satisfiable. If so, we can construct a satisfying assignment one variable at a time. Consider the following example:

$$F = (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_3 \vee \bar{x}_1) \equiv (x_1 \rightarrow x_2) \wedge (x_2 \rightarrow x_3) \wedge (x_3 \rightarrow \bar{x}_1)$$

Now since F is satisfiable, there must be some way to set (say) x_1 to either TRUE or FALSE so that the resulting formula still is satisfiable.

If we set x_1 to TRUE, then the resulting formula $F' = F|_{x_1 = TRUE}$ will become FALSE so it must be that x_1 is FALSE in any satisfying assignment.

How would we know that $F' = F|_{x_1 = FLASE}$ is satisfiable?

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How would we know that $F' = F|_{x_1 = FLASE}$ is satisfiable? We would again use the decision procedure SAT applied to F'. We would continue this way to see how to set x_2, x_3. In this example, x_2 can be set TRUE or FALSE and we would just choose one value. In general, a formula can have many satisfying assignments.

I know some (many?) students may find this to be difficult material as you would not have seen it before. Please ask questions

I do think this material is fundamental to computer science (as a discipline) and computing (in terms of its impact).

Some ideas are great ideas even when we are not that aware of them. I argued that this was the case with respect to Turing's work and the von Neumann model.

The concept of *NP* completeness is something that algorithm designers may or may not think of routinely but at some level of understanding we do need to know that common (say optimization) problems cannot be solved effciciently for all input instances.

I mentioned that there have been many surprises in complexity theory so I again emphasize that a conjecture may guide our thinking but we always have to be aware of what has and has not been proven.

Can randomization help?

We should note that there are many other fundamental questions in complexity theory (in addition to the P vs NP question). One such question is can randomization help.

Consider the following problem: We are *implicitly given* two multivariate polynomials $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$. For example, the polynomials might be the result of a polynomial time computation using the arithmetic operations $+, -, *$. Or p and q might be the determinants of $n \times n$ matrices with entries that are linear functions of the $\{x_i\}$.

The *polynomial equivalence question* whether or not $p \equiv q$ as polynomials; that is, does $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ for all values of the $\{x_i\}$. Lets say that the x_i are all integers or rationals.
Note that this is the same as asking whether or not $p - q \equiv \mathbf{0}$ where $\mathbf{0}$ is the zero polynomial.

How would you solve the *identically zero* question for a univariate polynomial (again given implicitly)?

Polynomial equivalence problem continued

Fact: A non zero univariate polynomial $p(x)$ of degree d has at most d distinct zeros. This means that if we evaluate $p(x)$ at say $t > d$ random points $r_1, \ldots r_t$, the probability that $p(r_i) = 0$ is at most $\frac{d}{t}$.

Schwartz-Zipple Lemma: This lemma extends the above fact to multivariate polynomials. That is,
If $p(x_1, \ldots, x_n)$ is a non zero polynomial of total degree d (with coeficients in a ring or field F like the integers or rationals) then $Prob[p(r_1, \ldots r_n) = 0] \leq \frac{d}{|S|}$ when the r_i are chosen randomly in a finite subset $S \subseteq F$.

Polynomial equaivalence and the class RP

So to test if p is identically zero, we take $|S|$ sufficienlty large (or do repeated independent trials with say $|S| = 2d$), and see if the evaluation returns a non-zero value. If $p(r_1, \ldots, r_n) = 0$, we will claim that $p \equiv \mathbf{0}$. The error in this claim will be at most $\frac{d}{|S|}$ and we will only make an error if $p \not\equiv \mathbf{0}$.

This is an example of a polynomial time randomized algorithm with 1-sided error (with say error at most $\frac{1}{2}$) and RP is the class of languages that have such an algorithm.

In fact the error can be as big as $1 - \frac{1}{n^k}$ for any fixed k as we can do polynomially many repeated trials to reduce the error probability using the fact that $(1 - 1/t)^t \to \frac{1}{e}$ as $t \to \infty$.

Open question: Is $RP = P$? As a specific example, is the polynomial equivalence problem in P?

RP and *BP*

Surprisingly, some prominent complexity theorists (but not everyone) believe $P = RP$. More generally, they believe $BPP = P$ where BPP is the class of languages that can be solved by a polynomial time randomized algorithm with 2-sided error (with probability of error at most $\frac{1}{2} - \frac{1}{n^k}$).

Like RP, we can amplify the probability of a correct answer by running a polynomial number of trials and taking the "majority vote" amongst the outcomes of the individual trials.

A langauge in RP can be formulated so that there are many certificates and hence $RP \subseteq NP$.

One final comment about the conjecture $P \neq NP$. While we strongly believe $P \neq NP$, all is not lost if $P \neq NP$. For example, while an optimization problem it can be NP-hard to compute an *optimal solution*, for many NP-hard problems there are efficient *approximately optimal* algorithms. And many natural problems have efficient algorithms when considering restricted classes of (or distributions over) instances that tend to occur naturally.

Complexity based cryptography and Public key encryption

In our discussion of cryptography, I am relying on CSC2426F graduate course notes by Charles Rackoff. See
http://www.cs.toronto.edu/ rackoff/2426f20/Cryptonotes.html

I may also be using web page notes by Paul Johnson. See
http://pajhome.org.uk/crypt/index.html

What is *complexity based cryptography?*
We are going to explore a counter-intuitive idea: Namely, the ability to use assumptions about what *cannot* be computed efficiently (i.e., negative results) to establish positive results for applications such as pseudo-random number generators, public key cryptography, digital signatures, secret sharing, and more. These applications all fall under the general topic of complexity based cryptography. Our focus will be on pseudo-random number generators (PRNG) and public key cryptography (PKC).

Randomization is necessary

Before we begin, we should note that *randomization* is almost always necessary for cryptography. This is not the first time we have encountered the need for randomization.

When have we used randomization before?

For various problems (say within NP), it seems that randomization is helpful but perhaps not provably so. That is, we do not know if RP and BPP are different from P or NP.

But there are applications where randomization is necessary

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- Simulating stochastic events
- Hashing
- Differential Privacy

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- Simulating stochastic events
- Hashing
- Differential Privacy
- And we can add cryoptography to this list

A secure shared secret key session

In this setting, two people called A and B (sometimes referred to as Alice and Bob) have been able to share *secret key* (e.g., a secret string of bits) and will use that secret key to communicate over an insecure channel. This insecure channel can be observed or perhaps even modified by an adversary.

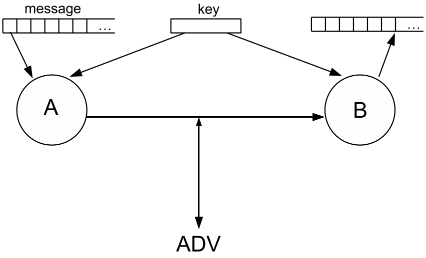

Figure: One-way communication. Figure taken from Rackoff notes

Figure: One-way communication. Figure taken from Rackoff notes

Shared-secret key session continued

An important consideration is how powerful is the adversary.
To do things reasonably carefully, we would probably need a full graduate course on cryptography. It is difficult enough to develop the main ideas even assuming that the adversary can only eavesdrop so lets make that assumption.

In a one-way session, A has an m bit message $M = M_1 M_2 \ldots M_m \in \{0,1\}^m$. (For simplicity, we are assuming that the message and the secret key have been reresented as a binary strings but this is not essential.) The message is called the *plain text*.

In the shared secret key setting we are assuming the A and B have agreed upon a secret key n bit key $K = K_1 K_2 \ldots K_n \in \{0,1\}^n$.

A will encode his message by a function $ENC : \{0,1\}^m \times \{0,1\}^n \to \{0,1\}^*$. Here we are using the $*$ to suggest that the encoded message length can depend on the plain text message. The encoded message is called *the cypher text*.

Shared-secret key session continued

B will decode the cypher text by a function
$DEC : \{0,1\}^* \times \{0,1\}^n \to \{0,1\}^m$.

What properties do we want from the exchange?

Shared-secret key session continued

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$DEC : \{0,1\}^* \times \{0,1\}^n \to \{0,1\}^m$.

What properties do we want from the exchange?

- **Privacy:** The adversary should not learn anyything "signifiicant" about the plain text. What does the adversary know in advance about the plain text?
- **Correctness:** B should be able to correctly decode the message. That is $DEC(ENC(M,K),K) = M$ for all M and K.

We might ask what is *perfectly secure session*? In the Rackoff notes #0, there are three equivalent definitions. Let's just use the first one. For a given plain text message M, a uniformly random key K induces a distribution D_M on the cypher texts. The session is perfectly private if the distribution D_M does not depend on M.

Is a perfectly secure session attainable?

End of Friday, November 27 class

We ended at slide 23.

We will finish up our discussion of complexity based cryptography next Friday, December 4.

The second and final quiz will take place, Wednesday, December 2 during the class time.

When is a perfectly secure session attainable?

Fact: A Perfectly secure session is attainable if and only if $|M| \leq |K|$.

When $|M| \leq |K|$, a *one-time pad* provides a single perfect secure session. A one-time pad is defined as follows:

$ENC(M, K) = E_1 E_2 \dots E_m$ where $E_i = M_i \oplus K_i$ for $1 \leq i \leq m$ and $DEC(E, K) = E_1 \oplus K_1, \dots, E_m \oplus K_m$.

Note that $\oplus$, the exclusive OR, flips a bit.

Warning: Never use an old key for a new purpose. For example, we cannot securely send two m bit messsages with the same m bit key.

So how are we going to continually generate random private keys (or long keys that can be partitioned into session keys) for different people to communicate? We cannot assume people can get together physically and even so how can they generate truly random strings of bits?

Complexity based assumptions; public key cryptography

The one-time pad does not need any assumptions and an adversary can have unlimited computational power and still cannot gain any information from a one-time pad. But as we noted, a one-time pad is not a very practical solution especially for frequent transactions in e-commerce.

The major application of *public key cryptography* is to enable key exchange. For public key cryptography (and almost all cryprographic applications) we will need complexity assumptions stronger than (but still widely accepted) $P \neq NP$. To make public key systems practical we will also need some sort of trusted public key infrastructre.

We will just discuss one well known public key system, *RSA*, which is based on the assumption that factoring large integers is hard even in some average sense (rather than worse case sense). This is a much stronger assumption than $P = NP$ since $P = NP$ would allow us to factor integers in polynomial time.

The basic idea of public key encryption

Public key encryption was introduced by Diffie and Hellman, and a particular method (RSA) was created by Rivest, Shamir and Adelman.

The basic idea is that in order for Alice (or anyone) to send Bob a message, Bob is going to create two related keys, a public key allowing Alice to send an encrypted mesasage to Bob, and a private key that allows Bob to decrypt Alice's message.

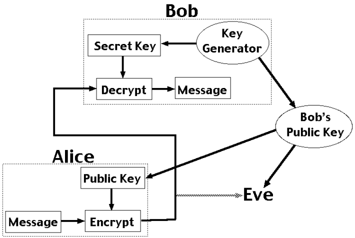

Figure: Diagram of public key encryption. Figure taken from Paul Johnston notes

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The RSA method

Bob wants to generate two keys, a public key e, N and a private key d. The claim is that it is hard on average to find d given e and N. Bob chooses $N = p \cdot q$ for two large primes p, q (which for defining "on average" may satisfy some constraint).

Bob will choose the public e such that $gcd(e, \phi(N)) = 1$ where $\phi(N) = \phi(pq) = (p-1)(q-1)$. $\phi(N)$ is called the Euler totient function which is equal to the number integers less than N that are relatively prime to N. $gcd(a, b) = 1$ means that a and b are relatively prime (i.e. have no common proper factors).

Alice encodes a message M by computing $M^e \bmod N$.

Hiding some mathematics, BOB can compute a d such that $de = 1 \bmod (p-1)(q-1)$ since Bob knows p and q. But without knowing p, q, finding d becomes computationally difficult.

Hiding some more mathematics, it will follow that $M^{de} = M \pmod N$ for any message M. That is, Bob decrypts a cypher text C by the function $C^d \bmod N$.

What mathematical facts do we need to know.

The main mathematical facts are :

1. There are sufficiently many prime numbers in any range so one can just randomly try to diffent numbers and test if they are prime.
2. $a^{\phi(N)} = 1 \bmod N$ for any a such that $gcd(a, N) = 1$ As a special case, $a^{p-1} = 1 \bmod p$ for any prime p and a not a multiple of p. So we have $M^{(p-1)(q-1)} = 1 \bmod N$.
3. If $gcd(a, b) = 1$ then there exists s and t such that $sa + tb = 1$. In the RSA algorithm, we can let $a = e$ and $b = (p-1)(q-1)$. Then s will become the d we need for decryption. That is $de + t(p-1)(q-1) = 1$.
4. It follows then that $M^{de} = M^{1-t(p-1)(q-1)} = M \cdot M^{-t(p-1)(q=1)} = M \bmod (p-1)(q-1)$.

What computational facts do we need to know?

1. The extended Euclidean algorithm can efficiently compute an s and t such that $sa + tb = gcd(a, b)$
2. a^k mod N can be computed efficiently for any a, k, N.
3. We can efficiently determine if a number p is prime.

In practice, public keys e are chosen to be reasonably small so that encryption can be made more efficient.

Note that we have been assuming that an adversary EVE (i.e., is just eavesdropping) and not changing messages. That is, EVE just wants to learn the message or something about the message. If EVE could change messages then EVE could pretend to be BOB. So one needs some sort of a public key infrastructure.

Note that if EVE knows that the message M was one a few possibilities, then EVE can try each of the possibilities; that is compute $M^e modN$ for each possible M to see what message was being sent. So here is where randomness can be used. We can pad or interspers random bits in the plain text M so that the message being sent becomes some one of many random messages M'.

WARNING: Real world cryptography is sophisticated

Complexity based cryptography requires careful consideration of the definitions and what precise assumptions are being made.

Complexity based cryptography has led to many important practical protocols and there are a number of theorems. Fortunatley, many complexity assumptions turn out to be equivalent.
In the Rackoff notes, the following theorem is stated as the fundamental theorem of cryptography. (To make this result precise, one needs precise definitions which we are omitting.)
Theorem: The following are equivalent:

- It is possible to do "computationally secure sssions"
- There exists pseudo-random generators; that is, create strings that computationally look random)
- There exist one way functions f; that is functions such that $f(x)$ is easy to compute but given $f(x)$ it is hard to find a z such that $f(z) = f(x)$
- There exist computationally secure digital signature schemes.

The discrete log function

RSA is based on the assume difficulty of factoring. Another assump;tion that is widely used in cryptography is the discrete log function. Again, we need some facts from number theory. Let p be a large prime.

- $\mathbb{Z}_p^*$ denotes the set of integers $\{1, 2, \ldots, p-1\}$ under the operations of $+, -, \cdot \mod p$ is a *field*. In particular, for every $a \in \mathbb{Z}_p^*$, there exists a $b \in \mathbb{Z}_p^*$ such that $a \cdot b = 1$; i.e., $b = a^{-1} \mod p$.
- Moroever, $\mathbb{Z}_p^*$ is *cyclic*. That is, there exists a $g \in \mathbb{Z}_p^*$ such that $\{1, g, g^2, g^3, \ldots g^{p-2}\} \mod p = \mathbb{Z}_p^*$. Recall, as a special case of the Euler totient function, $a^{p-1} = 1 \mod p$.

The assumption is that given $(g, p, g^x \bmod p)$, it is computationally difficult to find x. This is another example (factoring can also be an example) of a *one-way function*.

A pseudo random generator

We started off our discussion of complexity based cryptpgraphy by noting that randomness is essential. We have also noted that it is not clear (or at what cost) one can obtain strings that "look like" truly random strings.

A pseudo random generator G is a *deterministic* function $G : \{0,1\}^k \rightarrow \{0,1\}^\ell$ for $\ell > k$. When ℓ is exponential in k, G is called a pseudo random function generator. For now, lets even see how to be able to have $\ell = k + 1$.

The random input string $s \in \{0,1\}^k$ is called the seed and the goal is that $r = G(s)$ should be "computationally indistinguishable" from a truly random string in $t = \{0,1\}^\ell$. This means that no polynomial time algorithm can distinguish between r and t with probability better than $\frac{1}{2} + \epsilon$ for any $\epsilon > 0$. (Here I am being sloppy about the quantification but hopefully the idea is clear.)

A pseudo random generator continued

On the previous slide there was a claim that having a pseudo random generator is equivalent to having a one-way function.

How can we use (for example, the assumption that the discrete log function is a one-way function) to construct a pseudo random generator with $\ell = k + 1$.

The Blum-Micali generator. Assumming the discrete log function is a one-way function then the following is a pseudo random generator:

Let x_0 be a random seed in $\mathbb{Z}_p^*$ by interpeting $(s_1, \ldots, s_k)_2$ as a binary number mod p. Let $x_{k+1} = g^{x_k}$ mod p. Define $s_{k+1} = 1$ if $x_k \leq \frac{p-1}{2}$.

Manual Blum won the Turing award for his contributions to cryptography and Silvio Micali (along with Shafira Goldwasser) won the Turing award for *interactive zero knowledge proofs*.