Great Ideas in Computing

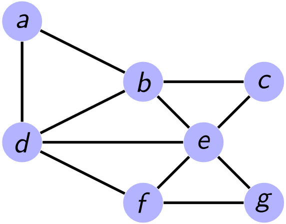
University of Toronto CSC196
Fall 2021

Week 11: Appendix

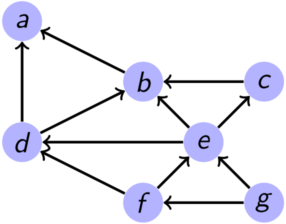
Appendix: Network (graph) definitions and examples

Graphs come in two varieties

1. undirected graphs ("graph" usually means an undirected graph.)



2. directed graphs (often called di-graphs).



Visualizing Networks as Graphs

- nodes: entities (people, countries, companies, organizations, ...)
- links (may be directed or weighted): relationship between entities
  - friendship, classmates, did business together, viewed the same web pages, ...
  - membership in a club, class, political party, ...

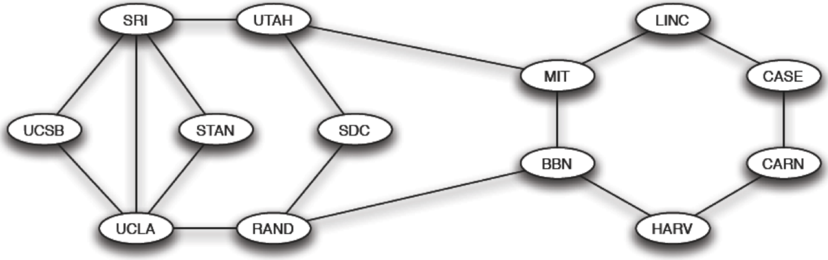


Figure: Internet: Dec. 1970 [E&K, Ch.2]

Adjacency matrix for graph induced by eastern sites) in 1970 internet graph: another way to represent a graph

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- This node induced subgraph (for the sites MIT = 1, LINC = 2, CASE = 3, CARN = 4, HARV = 5, BBN = 6) is a 6 node regular graph of degree 2. It is a simple graph in that there are no self-loops or multiple edges.
- Note that the adjacency matrix of an (undirected) simple graph is a symmetric matrix (i.e. $A_{i,j} = A_{j,i}$) with $\{0,1\}$ entries.
- To specify distances, we would need to give weights to the edges to represent the distances.

The matrix A^2 where $A = A(G)$

Consider squaring the previous matrix $A = A(G)$. That is, $A^2 = A * A$.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Draw a visualization of the graph represented by A^2. If we let $c_{i,j}$ be the i, j entry in A^2, can you desribe the meaning of $c_{i,j}$?

The matrix $B = A + I$

Consider the 6×6 identity matrix $I = (\iota_{i,j})$. That is, $\iota_{i,i} = 1$ for $1 \leq i \leq 6$ and $\iota_{i,j} = 0$ for $1 \leq i, j \leq 6$ and $i \neq j$.

Let $B = A + I$ (as above). That is, $b_{i,j} = a_{i,j} + \iota_{i,j}$ for all i, j. We have

$$B(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that now the matrix B has self loops and hence is not a simple graph.

Kidney Exchange: Swap Cycles

- Live kidney donation common in N.A. to get around waiting list problems: donor-recipient pairs are nodes and links are directed.
- Exchange: supports willing pairs who are incompatible
  1. allows multiway-exchange
  2. supported by sophisticated algorithms to find matches

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- Live kidney donation common in N.A. to get around waiting list problems: donor-recipient pairs are nodes and links are directed.
- Exchange: supports willing pairs who are incompatible
  1. allows multiway-exchange
  2. supported by sophisticated algorithms to find matches
- But what if someone reneges? ⇒ require simultaneous transplantation! Non-cyclic paths can be started by an altruistic donor!

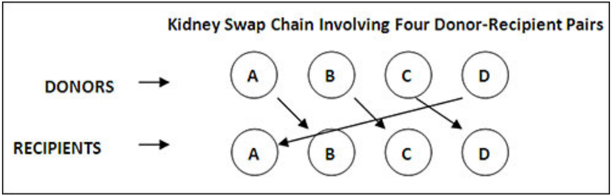
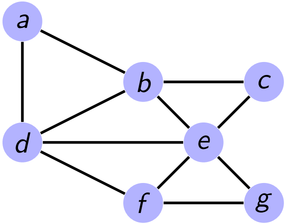
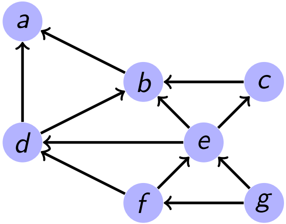


Figure: Dartmouth-Hitchcock Medical Center, NH, 2010

Recall: undirected graphs vs. directed graphs





More definitions and terminology

- In order to refer to the nodes and edges of a graph, we define graph $G = (V, E)$, where
  - V is the set of nodes (often called vertices)
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- Undirected graph: an edge (u, v) is an unordered pair of nodes.

- Directed graph: a directed edge (u, v) is an ordered pair of nodes $\langle u, v \rangle$.
  - However, we usually know when we have a directed graph and just write (u, v).

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- A path between two nodes, say u and v is a sequence of nodes, say $u_1, u_2, \ldots, u_k$, where for every $1 \leq i \leq k-1$,
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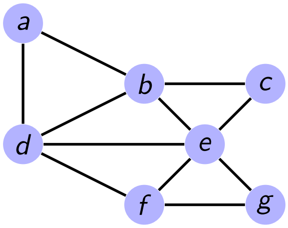
- The length of a path is the number of edges on that path.
- A graph is a connected if there is a path between every pair of nodes. For example, the following graph is connected.



Romantic Relationships [Bearman et al, 2004]

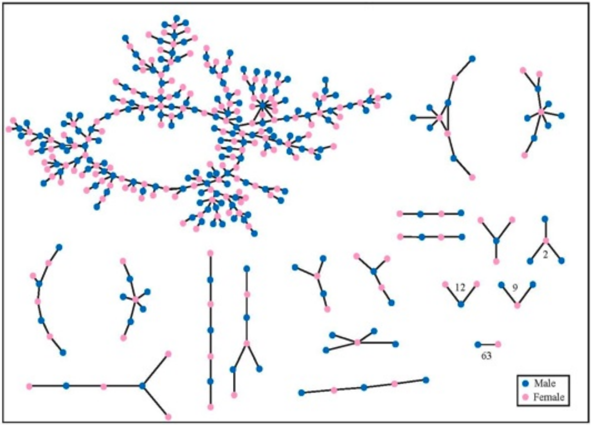
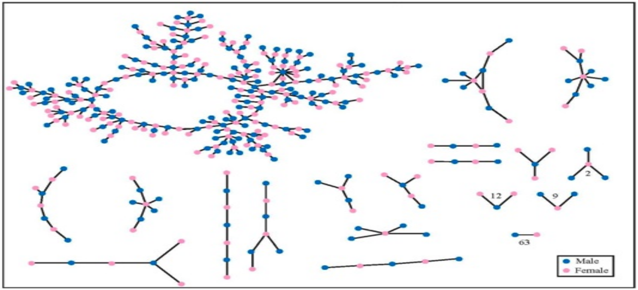


Figure: Dating network in US high school over 18 months.

- Illustrates common "structural" properties of many networks
- What predictions could you use this for?

More basic definitions



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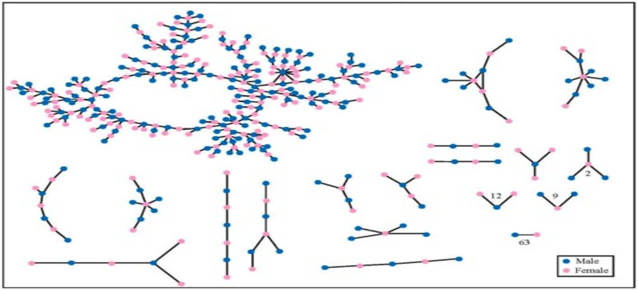

Observation

Many connected components including one "giant component"

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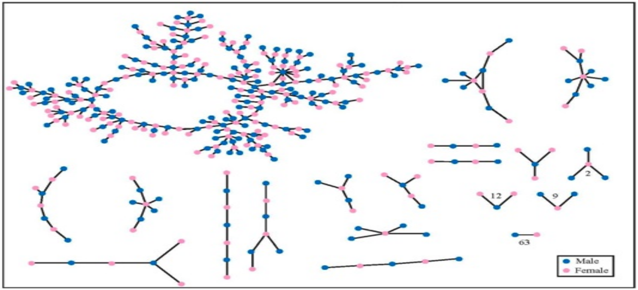


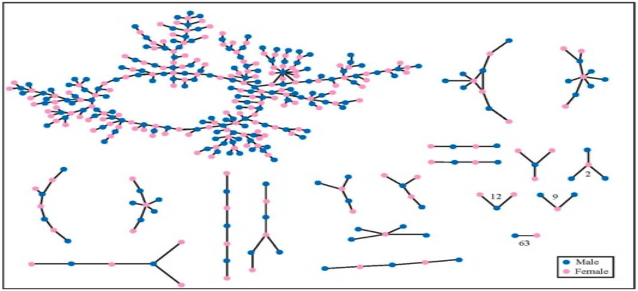
Observation

Many connected components including one "giant component"

- We will use this same graph to illustrate some other basic concepts.
- A cycle is path $u_1, u_2, \ldots, u_k$ such that $u_1 = u_k$; that is, the path starts and ends at the same node.

Simple paths and simple cycles

- Usually only consider simple paths and simple cycles: no repeated nodes (other than the start and end nodes in a simple cycle.)



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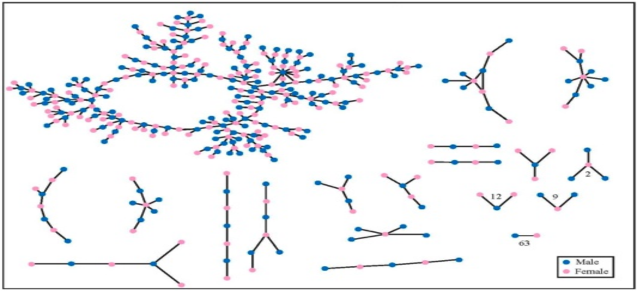

Observation

- There is one big simple cycle and (as far as I can see) three small simple cycles in the "giant component".
- Only one other connected component has a cycle: a triangle having three nodes. Note: this graph is "almost" bipartite and "almost" acyclic.

Example of an acyclic bipartite graph

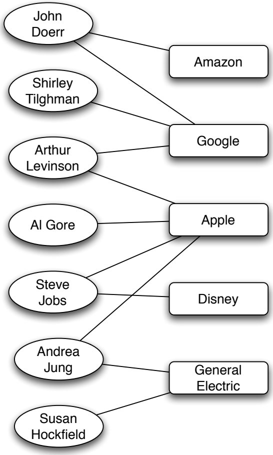

Figure: [E&K, Fig 4.4] One type of affiliation network that has been widely studied is the memberships of people on corporate boards of directors. A very small portion of this network (as of mid-2009) is shown here.

Florentine marriages and “centrality”

- Medici connected to more families, but not by much
- More importantly: lie between most pairs of families
  - shortest paths between two families: coordination, communication
  - Medici lie on 52% of all shortest paths; Guadagni 25%; Strozzi 10%

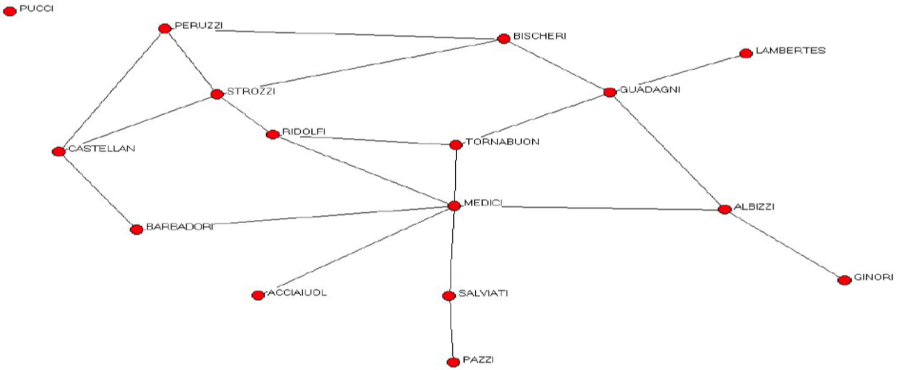

Figure: see [Jackson, Ch 1]

Breadth first search and path lengths [E&K, Fig 2.8]

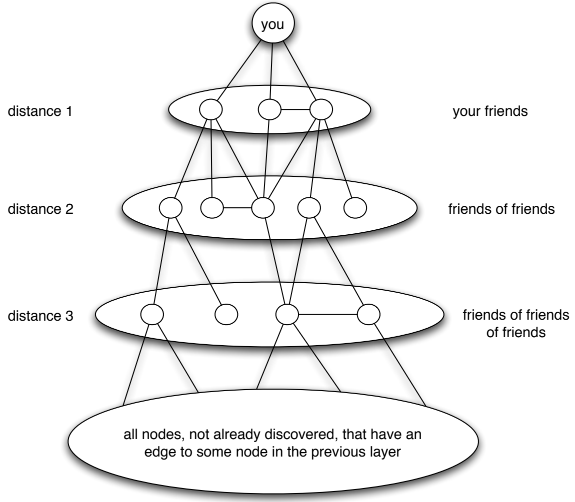

Figure: Breadth-first search discovers distances to nodes one “layer” at a time. Each layer is built of nodes adjacent to at least one node in the previous layer.

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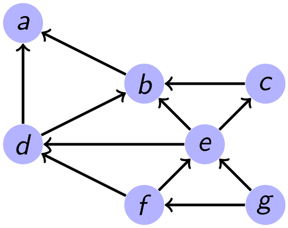
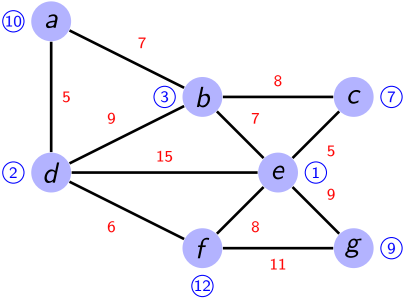
  - However, it is usually clear from context if we are discussing undirected or directed graphs and in both cases most people just write (u, v).
- We now have directed paths and directed cycles. Instead of connected components, we have strongly connected components.



Weighted graphs

- We will often consider weighted graphs. Lets consider a (directed or undirected) graph $G = (V, E)$. Example:



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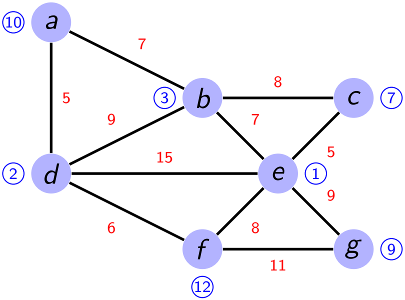

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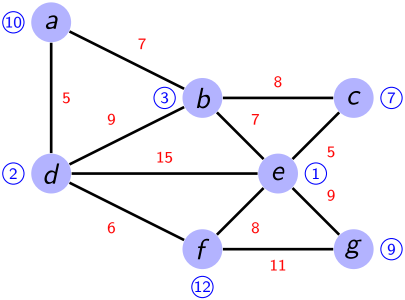

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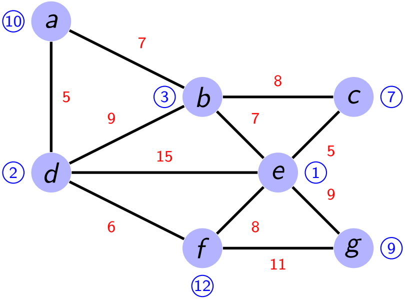

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- For example, in a social network whose nodes represent people, the weight $w(v)$ of node v might indicate the importance of this person.
- The weight $w(e)$ of edge e might reflect the strength of a friendship.

Edge weighted graphs

- When considering edge weighted graphs, we often have edge weights $w(e) = w(u, v)$ which are non negative (with $w(e) = 0$ or $w(e) = \infty$ meaning no edge depending on the context).
- In some cases, weights can be either positive or negative. A positive (resp. negative) weight reflects the intensity of connection (resp. repulsion) between two nodes (with $w(e) = 0$ being a neutral relation).
- Sometimes (as in Chapter 3) we will only have a qualitative (rather than quantitative) weight, to reflect a strong or weak relation (tie).
- Analogous to shortest paths in an unweighted graph, we often wish to compute least cost paths, where the cost of a path is the sum of weights of edges in the path.