Solutions

NAME (PRINT): ____
Last/Surname First /Given Name

STUDENT #: ______________________ SIGNATURE: ________________

UNIVERSITY OF TORONTO MISSISSAUGA
APRIL 2012 FINAL EXAMINATION
CSC338H5S
Numerical Methods
Anthony Bonner
Duration - 2 hours
Aids: The course textbook (Heath), and
one double-sided Letter (8-1/2 x 11) sheet of handwritten notes

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Please note, you ***CANNOT*** *petition to* ***re-write*** *an examination once the exam has begun.*

Write your answers on the examination sheet in the spaces provided. You may use the backs of pages if necessary. Concise, well-written answers will receive more points than long, rambling ones. Unless stated otherwise, all answers should be justified.

If you do not know the answer to a question, and you write "I don't know", you will receive 20% of the marks of that question. If you just leave a question blank with no such statement, you get 0 marks for that question.

This page is for marking purposes only

Question	Value	Score
1	16	
2	7	
3	10	
4	11	
5	10	
6	15	
7	8	
8	13	
Total	90	

1. (16 points total) True or False.

   For each of the following statements, state whether it is true or false, without giving a explanation (2 points each):

   (a) There are arbitrarily many different mathematical functions that interpolate a given set of data points.

   T

   (b) Points that minimize a nonlinear function are inherently less accurately determined than points for which a nonlinear function has a zero value.

   T

   (c) Newton's method for solving nonlinear equations is an example of a fixed-point iteration scheme.

   T

   (d) The eigenvalues of a matrix are not necessarily all distinct.

   T

   (e) At the solution to a linear least squares problem $Ax \approx b$, the residual $r = b - Ax$ is orthogonal to $span(A)$.

   T

   (f) An underdetermined system of linear equations $Ax = b$, where A is an $m \times n$ matrix with $m < n$, always has a solution.

   F

   (g) Using higher-precision arithmetic will make an ill-conditioned problem better conditioned.

   F

   (h) Every $n \times n$ matrix has n linearly-independent eigenvectors.

   F

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2. (7 points total) Approximation and Computer Arithmetic.

Consider the expression

$$\frac{1}{1-x} - \frac{1}{1+x}$$

assuming $x \neq \pm 1$.

(a) (2 points) For what range of values of x is it difficult to compute this expression accurately in floating-point arithmetic?

For $x \approx 0$, since the $\frac{1}{1-x} \approx \frac{1}{1+x} \approx 1$,

so we are subtracting two quanties of almost equal value, leading to large cancellation error.

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(b) (5 points) Give an equivalent expression such that, for the range of x in part (a), the computation is more accurate in floating-point arithmetic.

$$\frac{1}{1-x} - \frac{1}{1+x} = \frac{(1+x) - (1-x)}{(1-x)(1+x)}$$

$$= \frac{2x}{(1-x)(1+x)}$$

This expression leads to no cancellation error for $x \approx 0$.

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3. (10 points total) Systems of Linear Equations.

Suppose A, B and C are non-singular $n \times n$ matrices, and b is an n-vector. How would you efficiently evaluate the following expression without computing any matrix inverses:

$$((B^{-1})^2 + 3C)\underbrace{(A^{-1} + 5I)b}_{y}$$

Compute $L_1 + U_1$, the LU factorization of A.

Compute $L_2 + U_2$, the LU factorization of B.

Solve $L_1 U_1 x = b$ by subst. $(\therefore x = (L_1 U_1)^{-1} b = A^{-1} b)$

Let $y = x + 5b$ $(\therefore y = (A^{-1} + 5I) b)$

Solve $L_2 U_2 z = y$ by subst. $(\therefore z = (L_2 U_2)^{-1} y = B^{-1} y$

Solve $L_2 U_2 u = z$ by subst $(\therefore u = (L_2 U_2)^{-1} z = (B^{-1}) z = (B^{-1})^2 y)$

Let $v = z + 3(Cy)$ $(\therefore v = (B^{-1})^2 y + 3Cy = ((B^{-1})^2 + 3C) y)$

return v

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4. (11 points total) Linear Least Squares.

Suppose you are using Householder transformations to compute the QR factorization of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

(a) (2 points) How many Householder transformations are required?

3

(b) (5 points) Specify the first Householder transformation. (*i.e.*, Give the vector v for the transformation.)

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha = -\sqrt{1^2+1^2+1^2+1^2} = -\sqrt{4} = -2$

$$\therefore v = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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(c) (2 points) What does the first column of A become as a result of applying the first Householder transformation?

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

using $\alpha = -2$ from part b.

(d) (2 points) What does the first column of A then become as a result of applying the second Householder transformation?

$$\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

since the second Householder transformation does not affect the first column.

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5. (10 points total) Nonlinear Equations.

(a) (5 points) Consider the following iterative method:

$$x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Show that it is mathematically equivalent to the secant method for solving the scalar non-linear equation $f(x) = 0$.

The secant method is given by

$$x_{k+1} = x_k - f(x_k) \cdot \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

$$= \frac{[f(\cancel{x_k}) - f(x_{k-1})]\, x_k - f(x_k)\, [\cancel{x_k} - x_{k-1}]}{f(x_k) - f(x_{k-1})}$$

$$= \frac{x_{k-1} \cdot f(x_k) - x_k \cdot f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

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(b) (5 points) When implemented in finite-precision floating-point arithmetic, what advantages or disadvantages does the formula given in part (a) have over the formula for the secant method given in class (and in the text)?

The formula in part (a) is less accurate because it involves dividing two quantities that are both close to 0, and which are themselves inaccurate since they each involve the subtraction of two quantities of almost equal value (zero).

In contrast, the formula in the text adds a small correction to the current estimate of the solution. ~~Only the high-order bits of~~ ∴ The solution will be highly accurate even if the correction is not.

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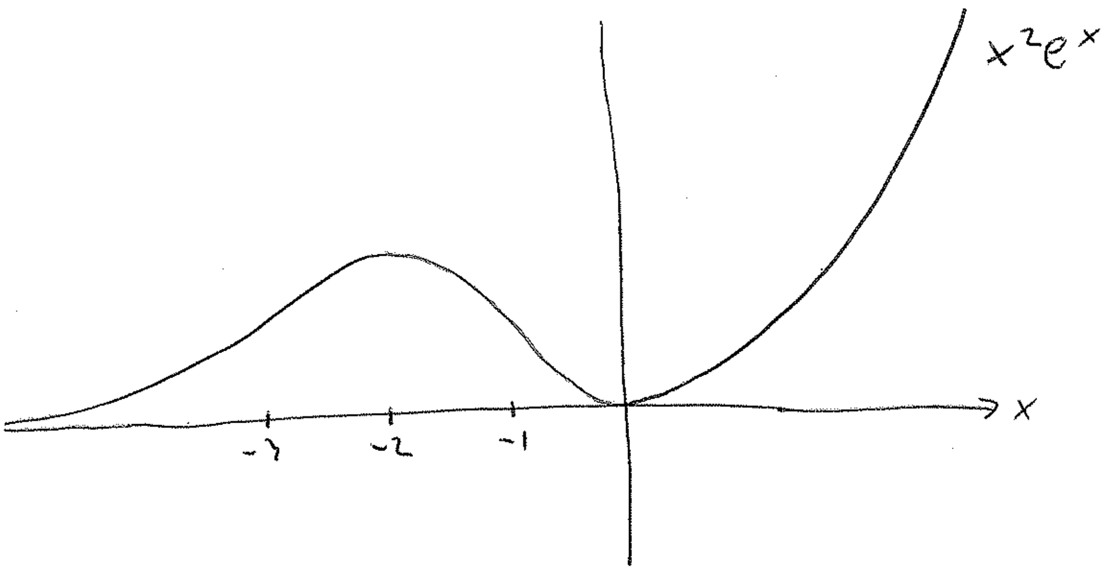
6. (15 points total) Optimization.

This question is about the function x^2e^x.

(a) (4 points) What are the critical points of the function?

$$0 = \frac{d\,x^2e^x}{dx} = 2xe^x + x^2e^x = x(2+x)e^x$$

iff $x = 0$ or $x = -2$ ← critical points.



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(b) (4 points) Characterize each critical point as a minimum, maximum or inflection point.

from the sketch of the function, obviously $x = 0$ is a minimum + $x = -2$ is a max.

To confirm this,

$$\frac{d^2 x^2 e^x}{dx} = \frac{d}{dx}(2x + x^2)e^x$$

$$= (2 + 2x)e^x + (2x + x^2)e^x$$

$$= (2 + 4x + x^2)e^x$$

$$\left.\frac{d^2 x^2 e^x}{dx}\right|_{x=0} = 2e^0 = 2 > 0 \quad \therefore x = 0 \text{ is a min.}$$

$$\left.\frac{d^2 x^2 e^x}{dx}\right|_{x=-2} = (2 - 8 + 4)e^{-2} = -2e^{-2} < 0$$

$\therefore x = -2$ is a max.

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(c) (2 points) Does the function have a global minimum or maximum?

$x^2 e^x \geq 0$

& $0^2 e^0 = 0$

$\therefore$ $x = 0$ is a global min

$x^2 e^x \to \infty$ as $x \to \infty$

$\therefore$ it has no global max.

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(d) (3 points) Assuming the function has a minimum, what is the Newton iteration for finding it?

$$x_{k+1} = x_k - f'(x_k) / f''(x_k)$$

$$\cancel{= x_k - \frac{(2x + x^2)e^x}{}}$$

$$\cancel{= x_k - \frac{x_k^2 e^{x_k}}{(2x_k + x_k^2)e^{x_k}}}$$

$$\cancel{= x_k - \frac{x_k}{2 + x_k}}$$

$$= x_k - \frac{(2x_k + x_k^2)e^{x_k}}{(2 + 4x_k + x_k^2)e^{x_k}}$$

$$= x_k - \frac{2x_k + x_k^2}{2 + 4x_k + x_k^2}$$

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(e) (2 points) If the starting guess for finding the minimum has an accuracy of 2 bits, how many iterations are required to obtain an accuracy of 53 bits?

$2 / \underbrace{4, 8, 16, 32, 64}_{5 \text{ iterations}}$ ← quadratic convergence

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7. (8 points total) Interpolation.

For each case below, write a formal algorithm for evaluating a polynomial at a given argument using Horner's nested evaluation scheme. Be sure to state clearly what the input to each algorithm is.

(a) (4 points) For a polynomial expressed in terms of the monomial basis

For the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$,

Horner's rule is $a_0 + x(a_1 + x(a_2 + \cdots x(a_{n-1} + a_n x) \cdots))$

input: $x, a_0, a_1 \cdots a_n$

let $p = a_n$

for $i = n$ to 1 by -1,

$p = a_{i-1} + x \cdot p$

end for

return(p)

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(b) (4 points) For a polynomial expressed in Newton form

For the polynomial

$$p(t) = a_0 + a_1(t-t_1) + a_2(t-t_1)(t-t_2) + \cdots + a_n(t-t_1)\cdots(t-t_n).$$

---

input = $t, t_1, \cdots, t_n, a_0, a_1, \cdots a_n$

$p = a_n$

for $i = n$ to 1 by -1,

  $p = a_{i-1} + (t-t_i) \cdot p$

end for

return (p)

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8. (13 points total) Eigenvalues.

(a) (4 points) Prove that the eigenvalues of a positive-definite matrix are all positive.

if $Ax = \lambda x$ + $x \neq 0$

then $x^T A x > 0$ since A is pos. def + $x \neq 0$

$\therefore x^T(\lambda x) > 0$

$\therefore \lambda(x^T x) > 0$

$\therefore \lambda \|x\|^2 > 0$

$\therefore \lambda > 0$ since $\|x\| > 0$ since $x \neq 0$.

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(b) (9 points) What are the eigenvalues and corresponding eigenvectors of the following matrix:

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

3 ~~eigen~~ Because the matrix is triangular, its diagonal elements are the eigenvalues: 1, 2, 3.

1 For $\lambda = 1$, ~~the~~ an eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

2 For $\lambda = 2$,

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore \begin{aligned} x + 2y - 4z &= 2x \\ \cancel{2y} + \cancel{z} &= \cancel{2y} \\ 3z &= 2z \end{aligned}$$

$\therefore z = 0$

& $2y = x$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector

check:

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

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This page is for answers and rough work.

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(3) For $\lambda = 3$,

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\therefore$
$$\begin{aligned} x + 2y - 4z &= 3x \\ 2y + z &= 3y \\ 3z &= 3z \end{aligned}$$

$\therefore y = z$

& $-2z = 2x$

$\therefore x = -z$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector

check:

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+2-4 \\ 2+1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

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This page is for answers and rough work.

Total marks = 90

END OF EXAM