Linear Algebra
Review
by Anthony Bonner

- Symmetric matrices
- Eigenvectors/eigenvalues
- Spectral decomposition
- Positive semi-definite matrices
Lemma 1: For real symmetric matrices, the eigenvectors of distinct eigenvalues are orthogonal.

Proof:
let \( x \) and \( y \) be eigenvectors of symmetric matrix \( A \), and let \( \lambda \) and \( \beta \) be their eigenvalues.

\[
\begin{align*}
A x &= \lambda x \\
A y &= \beta y
\end{align*}
\]

\[
\begin{align*}
y^T A x &= \lambda y^T x \\
x^T A y &= \beta x^T y
\end{align*}
\]
\[
\begin{align*}
\therefore \quad & \beta y^\top x \\
= \quad & (\beta x^\top y)^\top \\
= \quad & (x^\top A y)^\top \\
= \quad & y^\top A^\top x \\
= \quad & y^\top A x \quad \text{since } A = A^\top \\
= \quad & \alpha y^\top x \quad \text{by } x \\
\therefore \quad & \beta y^\top x = \alpha y^\top x \\
\therefore \quad & (\beta - \alpha) y^\top x = 0 \\
\text{If the eigenvalues are distinct, then } \beta \neq \alpha \\
\therefore \quad & y^\top x = 0 \\
\therefore \quad & y \perp x \\
\text{\underline{qed}}
\end{align*}
\]
Corollary 2: an $n \times n$ symmetric matrix has at most $n$ distinct eigenvalues, since it can have at most $n$ orthogonal eigenvectors.

Claim (hard).

An $n \times n$ symmetric matrix has exactly $n$ orthogonal eigenvectors.
Note: If \( x \) is an eigenvector of matrix \( A \), then so is \( ax \) for all \( a \in \mathbb{R}, a \neq 0 \), since

\[
A x = \lambda x
\]

iff \( a (Ax) = a(\lambda x) \)

iff \( A(ax) = \lambda (ax) \)

so, we can choose the eigenvectors of a matrix to have any length we want.
Theorem 3: Let $A$ be a real symmetric matrix. Then

$$A = O D O^T$$

where

- $O = (o_1, \ldots, o_n)$
- $o_i = \lambda_i o_i$  
- $\| o_i \| = 1$
- $o_i \perp o_j$, $i \neq j$

the $o_i$ are orthonormal eigenvectors of $A$.

- $D$ is a diagonal matrix whose $i$th diagonal entry is $\lambda_i$, the $i$th eigenvalue of $A$. 
Proof

Let $\sigma_1, \ldots, \sigma_n$ be the $n$ orthogonal eigenvectors of $A$.

\[ A \sigma_i = \lambda_i \sigma_i \]

$\sigma_i \perp \sigma_j$ for $i \neq j$.

Also, we can assume that $\| \sigma_i \| = 1$.

Let $0 = (0_1, \ldots, 0_n)$

\[ O^T O = I \text{ since } (O^T O)_{ij} = \sigma_i^T \sigma_j \]

\[ O^{-1} = O^T \]

(i.e., $O$ is an orthogonal matrix).

\[ A \sigma_i = \lambda_i \sigma_i \quad \text{For } i = 1 \ldots n \]
\[ (A \sigma_1, \ldots, A \sigma_n) = (\lambda_1 \sigma_1, \ldots, \lambda_n \sigma_n) \]
\[ A (\sigma_1 \ldots \sigma_n) = (\sigma_1 \ldots \sigma_n) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix} \]

\[ \therefore A O = O D \]
\[ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix} \]
\[ A = O D O^{-1} = O D O^T \]
Corollary 4:
The determinant of a real symmetric matrix is the product of its eigenvalues.

Proof:
\[ \det A = \det (O D O^T) \]
\[ = (\det O) \cdot (\det D) \cdot (\det O^T) \]
\[ = (\det O) \cdot (\det O^T) \cdot (\det D) \]
\[ = \det (O O^T) \cdot \det D \]
\[ = \det I \cdot \det D \]
\[ = \det D \]
\[ = \lambda_1 \ldots \lambda_n \quad \text{since } D \text{ is diagonal.} \]
\[ \text{q.e.d.} \]
Lemma 5:

A diagonal matrix, $D$, with non-negative diagonal entries is positive semi-definite, i.e., $z^T D z \geq 0 \quad \forall z \in \mathbb{R}^n$.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of $D$, and let $z = (z_1, \ldots, z_n)^T$

$$z^T D z = \sum_{i=1}^{n} z_i^2 \lambda_i$$

$$\geq 0 \quad \text{since} \quad \lambda_i \geq 0.$$
Corollary 6:

If $A$ is a real symmetric matrix
then it is positive semi-definite
iff all its eigenvalues are non-negative.

Proof: $A = O D O^T$

if direction:
If all the eigenvalues of $A$ are non-negative, then the diagonal entries of $D$ are non-negative, so $D$ is positive semi-definite.
\[ z^T A z = z^T D O^T z \]
\[ = (O^T z)^T D (O^T z) \]
\[ \geq 0 \]
\[ \therefore A \text{ is positive semi-def.} \]

\underline{only if direction:}

Suppose \( A \) is positive semi-def.
Let \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \)
\( i \)th position

Let \( z_i = 0 e_i \)
\[ \therefore e_i = O^T z_i \]

Let \( \lambda_i \) be the \( i \)th eigenvalue of \( A \).
\( \lambda_i \) is the \( i \text{th} \) diagonal element of \( D \).

\[
\lambda_i = e_i^T D e_i \\
= (O^T z_i)^T D (O^T z_i) \\
= z_i^T O D O^T z_i \\
= z_i^T A z_i \\
> 0 \text{ since } A \text{ is positive semi-definite.}
\]

\( \text{q.e.d.} \)
Corollary 7.

If $A$ is a symmetric, positive semi-definite matrix, then $A = B B^T$ for some matrix $B$.

Proof. $A = O D O^T$ where the diagonal entries of $O$ are non-negative.

Let $\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_n} \end{pmatrix}$ where $\lambda_i > 0$

$\sqrt{D} \cdot \sqrt{D} = D$
\[ A = 0 \sqrt{\mathbf{D}} \sqrt{\mathbf{D}} \ 0^T \]
\[ = 0 \sqrt{\mathbf{D}} (\sqrt{\mathbf{D}})^T 0^T \]
\[ = (0 \sqrt{\mathbf{D}}) (0 \sqrt{\mathbf{D}})^T \]
\[ = B B^T \]

where \( B = 0 \sqrt{\mathbf{D}} \)

qed.
Corollary 8.

\[ A \text{ is a symmetric, positive semi-def matrix} \iff \]
\[ A = B B^T \text{ for some matrix } B. \]

Proof.

Only-if direction.
This is corollary 7.

If direction.

If \[ A = B B^T \]
then \[ A^T = (B B^T)^T = (B^T)^T B^T = B B^T = A. \]
\[ A \text{ is symmetric.} \]

Also,

\[
\begin{align*}
2^T A z &= 2^T B B^T z \\
&= (B^T z)^T (B^T z) \\
&= \| B^T z \|^2 \\
&\geq 0
\end{align*}
\]

\[ \therefore A \text{ is pos. semi-def.} \]

\[ \square \text{ed.} \]
Corollary 9.

If $A$ is a symmetric, positive semi-definite matrix, then $A = B^2$ for some positive symmetric, positive semi-definite matrix $B$. (Note: we often say $B = \sqrt{A}$).

Proof: $A = ODO^T$

where $O^TO = I$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$ is diagonal

and $\lambda_i > 0$ (by corollary 6)

Let $C = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_n} \end{bmatrix}$

$CC^T \Rightarrow C^2 = D$
\[ A = O D O^T = O C^2 O^T = O C (O^T O) C O^T \text{ since } O^T O = I = (O C O^T) (O C O^T) = B^2 \text{ where } B = O C O^T \]

Note that \( B \) is symmetric and positive semi-definite (since the diagonal entries of \( C \) are non-negative).

\[ \text{QED.} \]
Let \( \mathbf{V} = (V_1, \ldots, V_m)^T \) where each \( V_i \) is a real-valued random variable with mean \( 0 \), i.e., \( E(V_i) = 0 \).

**Definition.** The covariance matrix of \( \mathbf{V} \), denoted \( \text{cov}(\mathbf{V}) \), is the matrix \( \mathbf{B} \) where \( B_{ij} = E(V_i \cdot V_j) \).

**Thm 10:** \( \text{cov}(\mathbf{V}) \) is symmetric and positive semi-definite.

**Proof.** Symmetry is easy:

\[
B_{ij} = E(V_i \cdot V_j) = E(V_j \cdot V_i) = B_{ji};
\]

\( B = B^T \).
To show positive semi-definiteness,

let \( z = (z_1, \ldots, z_m)^T \in \mathbb{R}^m \).

\[
\begin{align*}
  z^T B z &= \sum_{i,j} z_i B_{i,j} z_j \\
             &= \sum_{i,j} z_i E(v_i, v_j) z_j \\
             &= E \left( \sum_{i,j} z_i v_i v_j z_j \right) \\
             &= E \left[ \left( \sum_i z_i v_i \right) \left( \sum_j z_j v_j \right) \right] \\
             &= E \left[ \left( \sum_i z_i v_i \right)^2 \right] \\
             &> 0
\end{align*}
\]

\( B \) is pos. semi-def. \( \blacksquare \).
Lemma 11: If \( A \) and \( B \) are symmetric, positive semi-definite matrices, then so is \( C \), where \( C_{ij} = A_{ij} B_{ij} \).

Proof: Let \( V \) and \( W \) be independent random vectors, where

\[
V = (v_1, \ldots, v_m)^T \sim \mathcal{N}(0, A)
\]
\[
W = (w_1, \ldots, w_m)^T \sim \mathcal{N}(0, B)
\]

i.e., \( V \) and \( W \) are normally distributed with mean 0 and

\[
A = \text{cov}(V) \quad \text{and} \quad A_{ij} = E(V_i V_j)
\]
\[
B = \text{cov}(W) \quad \text{and} \quad B_{ij} = E(W_i W_j)
\]

Therefore, \( C = \text{cov}(V, W) \) is also normally distributed with mean 0 and

\[
C_{ij} = E(V_i W_j)
\]

as required.
Let \( X = (X_1, \ldots, X_m)^T \) where \( X_i = U_i \cdot W_i \).

\[
E(X_i) = E(U_i \cdot W_i) = E(U_i) \cdot E(W_i) \quad \text{by independence}
\]

\[
= 0 \cdot 0 = 0
\]

\[
[cov(X)]_{ij}
\]

\[
= E(X_i X_j) \quad \text{by definition}
\]

\[
= E[(U_i W_i)(U_j W_j)]
\]

\[
= E[(U_i U_j) \cdot (W_i W_j)]
\]

\[
= E(U_i U_j) \cdot E(W_i W_j) \quad \text{by independence}
\]

\[
= A_{ij} B_{ij} \quad \text{by *}
\]

\[
= C_{ij} \quad \text{by definition}
\]

\[
C = cov(X)
\]

\[ C \text{ is sym. & pos. semi-def.} \quad \text{by Thm 10.}\]