Linear Algebra Review

by Anthony Bonner

- Symmetric matrices
- Eigenvectors / eigenvalues
- Spectral decomposition
- Positive semi-definite matrices

Lemma 1: For real symmetric matrices, the eigenvectors of distinct eigenvalues are orthogonal.

Proof:

let x & y be eigenvectors of symmetric matrix A, & let α & β be their eigen values

$\therefore$ $Ax = \alpha x$

& $Ay = \beta y$

$\therefore$ $y^T A x = \alpha y^T x$ $\quad *$

& $x^T A y = \beta x^T y$

$\therefore \beta y^T x$

$= (\beta x^T y)^T$

$= (x^T A y)^T$

$= y^T A^T x$

$= y^T A x$ since $A = A^T$

$= \alpha y^T x$ by $*$

$\therefore \beta y^T x = \alpha y^T x$

$\therefore (\beta - \alpha) y^T x = 0$

If the eigenvalues are distinct, then $\beta \neq \alpha$

$\therefore y^T x = 0$

ie $y \perp x$

qed.

<u>Corollary 2</u>: an $n \times n$ symmetric matrix has at most n distinct eigenvalues, since it can have at most n orthogonal eigenvectors.

<u>Claim</u> (hard).

An $n \times n$ symmetrix matrix has <u>exactly</u> n orthogonal eigenvectors.

Note: If x is an eigenvector of matrix A, then so is ax, for all $a \in \mathbb{R}$, $a \neq 0$, since

$$Ax = \lambda x$$

$$\text{iff} \quad a(Ax) = a(\lambda x)$$

$$\text{iff} \quad A(ax) = \lambda(ax)$$

So, we can choose the eigenvectors of a matrix to have any length we want.

<u>Theorem 3</u>: Let A be a real symmetric matrix. Then

$$A = ODO^T$$

where

- $O = (o_1 \cdots o_n)$
- $A o_i = \lambda_i o_i$
- $\|o_i\| = 1$
- $o_i \perp o_j \qquad i \neq j$

the o_i are orthonormal eigenvectors of A.

- D is a diagonal matrix whose i^{th} diagonal entry is λ_i, the i^{th} eigenvalue of A.

Proof

Let $o_1 \cdots o_n$ be the n orthogonal eigenvectors of A.

$\therefore \quad A o_i = \lambda_i o_i$

& $\quad o_i \perp o_j \quad$ for $i \neq j$

Also, we can assume that $\|o_i\| = 1$.

Let $O = (o_1 \cdots o_n)$

$\therefore \; O^T O = I \quad$ since $(O^T O)_{ij} = o_i^T o_j$

$\therefore \; O^{-1} = O^T$

(i.e, O is an orthogonal matrix).

$$A\sigma_i = \lambda_i \sigma_i \qquad \text{For} \quad i = 1 \cdots n$$

$$\therefore (A\sigma_1, \cdots, A\sigma_n) = (\lambda_1 \sigma_1, \cdots, \lambda_n \sigma_n)$$

$$\therefore A(\sigma_1 \cdots \sigma_n) = (\sigma_1 \cdots \sigma_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

ie, $AO = OD$

$\therefore \cancel{AO}$

$$\therefore A = ODO^{-1} = ODO^T$$

qed.

Corollary 4:

The determinant of a real symmetric matrix is the product of its eigenvalues.

Proof.

$$\begin{aligned}
\text{Det } A &= \det(ODO^T) \\
&= (\det O)\cdot(\det D)\cdot(\det O^T) \\
&= (\det O)\cdot(\det O^T)\cdot(\det D) \\
&= \det(OO^T)\cdot \det D \\
&= \det I \cdot \det D \\
&= \det D \\
&= \lambda_1 \cdots \lambda_n
\end{aligned}$$

since D is diagonal.

q.e.d.

Lemma 5:

A diagonal matrix, D, with non-negative diagonal entries is positive semi-definite, ie, $z^T D z \geq 0 \quad \forall z \in \mathbb{R}^n$.

Proof. Let $\lambda_1 \cdots \lambda_n$ be the diagonal entries of D, & let $z = (z_1 \cdots z_n)^T$

$$\therefore z^T D z = \sum_{i=1}^{n} z_i^2 \lambda_i$$

$$\geq 0 \quad \text{since } \lambda_i \geq 0.$$

Corollary 6:

If A is a real symmetric matrix, then it is positive semi-definite iff all its eigenvalues are non-negative.

Proof. $A = ODO^T$

if direction:

If all the eigenvalues of A are non-negative, then the diagonal entries of D are non-negative, so D is positive semi-definite.

$$\therefore\ z^T A z = z^T O D O^T z$$
$$= (O^T z) D (O^T z)$$
$$\geq 0$$

$\therefore$ A is positive semi-def.

<u>only if direction:</u>

Suppose A is positive semi-def.

Let $e_i = (0 \cdots 0 1 0 \cdots 0)^T$

ith position

Let $z_i = O e_i$

$\therefore$ $e_i = O^T z_i$

Let λ_i be the $i^{\underline{th}}$ eigenvalue of A.

$\therefore$ λ_i is the ith diagonal element of D.

$$\begin{aligned}\therefore\ \lambda_i &= e_i^T D e_i \\ &= (O^T z_i)^T D (O^T z_i) \\ &= z_i^T O D O^T z_i \\ &= z_i^T A z_i \\ &\geq 0 \qquad \text{since } A \text{ is positive semi-definite.}\end{aligned}$$

qed.

Corollary 7.

If A is a symmetric, positive semi-definite matrix, then $A = BB^T$ for some matrix B.

Proof. $A = ODO^T$ where the diagonal entries of D are non-negative.

i.e., $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ where $\lambda_i \geq 0$

Let $\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$

note: $\sqrt{D} \cdot \sqrt{D} = D$

$$\therefore\ A = O\sqrt{D}\sqrt{D}\,O^T$$

$$= O\sqrt{D}\,(\sqrt{D})^T O^T$$

$$= (O\sqrt{D})(O\sqrt{D})^T$$

$$= BB^T$$

where $B = O\sqrt{D}$

qed.

Corollary 8.

A is a symmetric, positive semi-def matrix iff $A = BB^T$ for some matrix B.

Proof.

only-if direction

This is corollary 7.

if direction

If $A = BB^T$

then $A^T = (BB^T)^T$

$$= (B^T)^T B^T$$

$$= BB^T = A$$

$\therefore$ A is symmetic.

Also,

$$
\begin{aligned}
z^T A z &= z^T B B^T z \\
&= (B^T z)^T (B^T z) \\
&= \| B^T z \|^2 \\
&\geq 0
\end{aligned}
$$

$\therefore$ A is pos. semi-def.

qed.