CSC410 tutorial: SAT for problem solving

Encoding problems to SAT problems

October 2, 2019
Motivation

• Boolean satisfiability problem (SAT):
  Given formula e.g. \((x \lor y \lor z) \land (\neg x \lor \neg y) \land (\neg y \lor \neg z)\),
  is there an assignment that makes the formula true?

• Efficient algorithm to solve SAT problems.

• Solver, e.g. Z3 (demo).
You have a problem to solve, e.g. need to write a Sudoku solver.

A 9x9 grid, with 81 variables: $0 < i, j \leq 9$, $x_{ij}$ is the value (digit from 1 to 9) of cell $i, j$.

**Constraints:** each row, column an block of 3x3 contains all the digits from 1 to 9. Some digits are already assigned.

* is it a boolean SAT problem? **No.**
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- **Solution 1**: design a specialized algorithm and **optimize** it.
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You have a problem to solve, e.g. need to write a Sudoku solver.

- is it a boolean SAT problem? No.
- **Solution 1:** design a specialized algorithm and optimize it.
- Solution 2: **encode it** into a SAT problem, use **off-the-shelf optimized solver**.
• **Original problem**: non-boolean variables and constraints.

• **SAT encoding**: boolean variables and clauses.
**SAT encoding**

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  - **variables**: find an interpretable boolean representation of the original variables.
    Ex: binary representation of integers.
SAT encoding

- **Original problem**: non-boolean variables and constraints.
- **SAT encoding**: boolean variables and clauses.
  - **variables**: find an interpretable boolean representation of the original variables.
    Ex: binary representation of integers.
  - **clauses**: express the constraints as a conjunction of clauses.
Sudoku

A 9x9 grid, with 81 variables:
0 < i, j ≤ 9, x_{ij} is the value (digit from 1 to 9) of cell i, j.

Constraints:

- each row contains all the digits from 1 to 9,
- each column contains all the digits from 1 to 9,
- each block of 3x3 contains all the digits from 1 to 9,
- initial grid specifies values of some cells.
A 9x9 grid, with 9x81 variables:
0 < i ≤ 9, 0 < j ≤ 9, 0 < k ≤ 9, \( b_{i,j,k} \) is true iff the value of cell \( i, j \) is \( k \).
Sudoku: SAT version

Constraints:

• each row contains all the digits:
  \[ \forall 0 < k \leq 9, 0 < i \leq 9, \sum_{j=1}^{9} b_{i,j,k} = 1 \]
Sudoku: SAT version

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Exactly one

Exactly one \( \{x_i\}_{i=1..n} \) is true: \( \sum_{i=1}^{n} x_i = 1 \).

\( \binom{N}{2} + 1 \) clauses:

- At least one: \( x_1 \lor x_2 \lor \ldots \lor x_n \).
- At most one (no pair with both \( x_i, x_j \) true):
  \( \neg x_i \lor \neg x_j, \) for \( i, j = 1..n \) and \( i \neq j \).
Sudoku : SAT version

Constraints:

• each column contains all the digits:
  \[
  \forall 0 < k \leq 9, \ 0 < j \leq 9, \ \sum_{i=1}^{9} b_{i,j,k} = 1
  \]
Sudoku: SAT version

Constraints:

- each block of 3x3:

\[
\forall 0 < k \leq 9, 0 < a, b \leq 3, \\
\sum_{r=1}^{3} \sum_{l=1}^{3} b_{3a+r,3b+l,k} = 1
\]
Sudoku: SAT version

Constraints:

- initial grid specifies values of some cells.