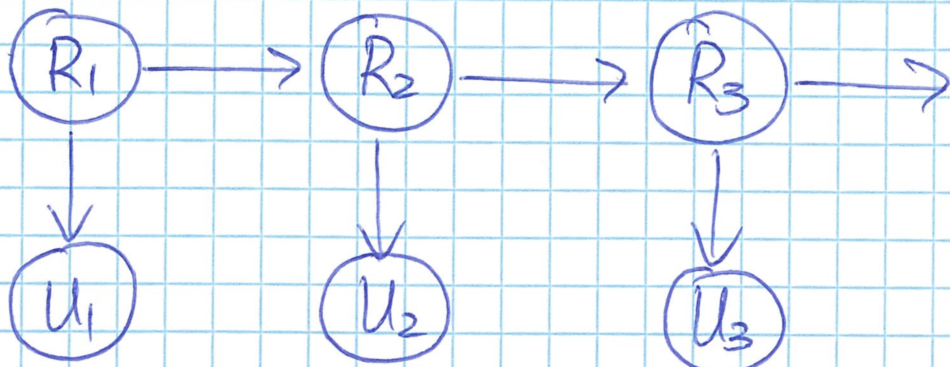
$P(R_1) = 0.5$

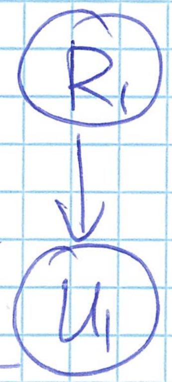
$P(R_t \mid R_{t-1}) = 0.7$ $\qquad$ $P(U_t \mid R_t) = 0.9$

$P(R_t \mid \neg R_{t-1}) = 0.3$ $\qquad$ $P(U_t \mid \neg R_t) = 0.2$



Filtering

$t = 1$ What is $P(\overset{=r_1}{R_1} \mid U_1)$? $U_1 = t$



$$P(\overset{=r_1}{R_1} \mid U_1) = \frac{P(U_1 \mid \overset{=r_1}{R_1})\, P(\overset{=r_1}{R_1})}{P(U_1)} \quad \text{Bayes' rule}$$

$$= \frac{P(U_1 \mid \overset{=r_1}{R_1})\, P(\overset{=r_1}{R_1})}{P(U_1 \mid \neg R_1) P(\neg R_1) + P(U_1 \mid R_1) P(R_1)}$$

$$= \alpha\, P(U_1 \mid \overset{=r_1}{R_1})\, P(\overset{=r_1}{R_1})$$

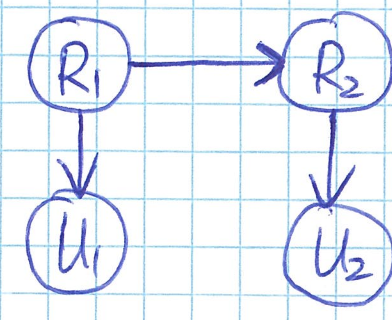
$$= \alpha \langle 0.9 * 0.5 ,\ 0.2 * 0.5 \rangle$$

$$= \alpha \langle 0.45 ,\ 0.1 \rangle$$

$$= \langle 0.818 ,\ 0.182 \rangle$$

$t=2 \quad U_1 = t, \; U_2 = t \quad$ *What is* $P(R_2 = r_2 \mid U_1 \wedge U_2)$?

$P(R_2 = r_2 \mid U_1 \wedge U_2)$



$$= \alpha P(U_2 \mid R_2 = r_2 \wedge U_1)\, P(R_2 = r_2 \mid U_1)$$

$P(R_2 = r_2 \mid U_1 \wedge U_2)$

$$= \frac{P(R_2 = r_2 \wedge U_1 \wedge U_2)}{P(U_1 \wedge U_2)} \qquad \alpha' = \frac{P(U_1)}{P(U_1 \wedge U_2)}$$

$$= \alpha P(R_2 = r_2 \wedge U_1 \wedge U_2)$$

$$= \alpha P(U_2 \mid R_2 = r_2 \wedge U_1)\, P(R_2 = r_2 \mid U_1)\, P(U_1)$$

$$= \alpha' P(U_2 \mid R_2 = r_2 \wedge U_1)\, P(R_2 = r_2 \mid U_1)$$

$$= \alpha P(U_2 \mid R_2 = r_2 \wedge U_1)\, P(R_2 = r_2 \mid U_1)$$

$$= \alpha P(U_2 \mid R_2 = r_2)\, P(R_2 = r_2 \mid U_1)$$ sensor Markov assumption.

$$= \alpha P(U_2 \mid R_2 = r_2) \sum_{r_1} P(R_2 = r_2 \mid R_1 = r_1 \wedge U_1)\, P(R_1 = r_1 \mid U_1)$$

$$P(R_2 = r_2 \mid U_1) = \frac{P(R_2 = r_2 \wedge U_1)}{P(U_1)}$$

$$= \frac{1}{P(U_1)} P(R_2 = r_2 \wedge U_1) = \frac{1}{P(U_1)} \sum_{r_1} P(R_2 = r_2 \wedge R_1 = r_1 \wedge U_1)$$

$$= \frac{1}{P(U_1)} \sum_{r_1} P(R_2 = r_2 \mid R_1 = r_1 \wedge U_1)\, P(R_1 = r_1 \mid U_1)\, P(U_1)$$

$$= \sum_{r_1} P(R_2 = r_2 \mid R_1 = r_1 \wedge U_1)\, P(R_1 = r_1 \mid U_1)$$

$$P(R_2=r_2 \mid U_1 \wedge U_2)$$

$$= \alpha P(U_2 \mid R_2=r_2) \sum_{r_1} P(R_2=r_2 \mid R_1=r_1 \wedge U_1) P(R_1=r_1 \mid U_1)$$

$$= \alpha P(U_2 \mid R_2=r_2) \sum_{r_1} P(R_2=r_2 \mid R_1=r_1) P(R_1=r_1 \mid U_1)$$

by the Markov assumption.

$$P(R_2=t \mid U_1 \wedge U_2)$$

$$= \alpha P(U_2 \mid R_2=t) \sum_{r_1} P(R_2=t \mid R_1=r_1) P(R_1=r_1 \mid U_1)$$

$$= \alpha\ 0.9 * (0.7 * 0.818 + 0.3 * 0.182)$$

$$= \alpha\ 0.9 * 0.6272$$

$$= \alpha\ 0.564$$

$$P(R_2=f \mid U_1 \wedge U_2)$$

$$= \alpha P(U_2 \mid R_2=f) \sum_{r_1} P(R_2=f \mid R_1=r_1) P(R_1=r_1 \mid U_1)$$

$$= \alpha\ 0.2 * (0.3 * 0.818 + 0.7 * 0.182)$$

$$= \alpha\ 0.2 * 0.3728$$

$$= \alpha\ 0.075$$

After normalizing, $\langle 0.564, 0.075 \rangle$ becomes $\langle 0.883, 0.117 \rangle$

$$x_t = \underline{\underline{P(\overset{=r_t}{R_t} \mid U_1 \wedge \cdots \wedge U_t)}}$$

$$= \alpha\, P(U_t \mid R_t = r_t \wedge U_1 \wedge \cdots \wedge U_{t-1})$$
$$* \, P(R_t = r_t \mid U_1 \wedge \cdots \wedge U_{t-1})$$

One-step Prediction.

$$= \alpha\, P(U_t \mid R_t = r_t)\, \underline{\underline{P(R_t \mid U_1 \wedge \cdots \wedge U_{t-1})}}$$

by sensor Markov assumption.

$$= \alpha\, P(U_t \mid R_t = r_t) *$$
$$\sum_{R_{t-1}} P(\overset{=r_t}{R_t} \mid R_{t-1} \wedge U_1 \wedge \cdots \wedge U_{t-1})\, P(R_{t-1} \mid U_1 \wedge \cdots \wedge U_{t-1})$$

$$= \alpha\, P(U_t \mid R_t = r_t) \sum_{R_{t-1}} P(\overset{=r_t}{R_t} \mid R_{t-1})\, \underline{\underline{P(R_{t-1} \mid U_1 \wedge \cdots \wedge U_{t-1})}}$$

by Markov assumption.

Observations:

- $P(R_1 = t \mid U_1) = 0.818$

  $P(R_1 = t \mid U_1 \wedge U_2) = 0.883$

- Forward Recursion for filtering.

  $P(R_1 = r_1 \mid U_1) \Rightarrow P(R_2 = r_2 \mid U_1 \wedge U_2)$

  $\Rightarrow P(R_3 = r_3 \mid U_1 \wedge U_2 \wedge U_3)$.

Prediction w/ Forward Recursion.

one-step

$$P(R_t \mid U_1 \wedge \cdots \wedge U_{t-1})$$

$$= \sum_{R_{t-1}} P(R_t \mid R_{t-1})\, P(R_{t-1} \mid U_1 \wedge \cdots \wedge U_{t-1})$$

general case

$$\underline{\underline{P(R_{t+k+1} \mid U_1 \wedge \cdots \wedge U_{t-1})}}$$

$$= \sum_{R_{t+k}} P(R_{t+k+1} \mid R_{t+k})\, \underline{\underline{P(R_{t+k} \mid U_1 \wedge \cdots \wedge U_{t-1})}}$$

Smoothing

$$P(R_k \mid U_1 \wedge \cdots \wedge U_t)$$

$$= P(R_k \mid U_1 \wedge \cdots \wedge U_k \wedge U_{k+1} \wedge \cdots \wedge U_t)$$

$$= \alpha P(R_k \mid U_1 \wedge \cdots \wedge U_k)\, P(U_{k+1} \wedge \cdots \wedge U_t \mid R_k \wedge U_1 \wedge \cdots \wedge U_k)$$

$$= \alpha \underbrace{P(R_k \mid U_1 \wedge \cdots \wedge U_k)}_{\substack{\text{filtering} \\ \text{forward recursion}}}\, \overbrace{P(U_{k+1} \wedge \cdots \wedge U_t \mid R_k)}^{\text{backward recursion}}$$

(using conditional independence)

Backward Recursion

$$\underline{\underline{P(U_{k+1} \wedge \cdots \wedge U_t \mid R_k)}}$$

$$= \sum_{R_{k+1}} P(U_{k+1} \mid R_{k+1})\, \underline{\underline{P(U_{k+2} \wedge \cdots \wedge U_t \mid R_{k+1})}}$$

$$* P(R_{k+1} \mid R_k)$$

Initialized with a vector of 1

when $k+1 > t$, $U_{k+1} \wedge \cdots \wedge U_t$ is an empty sequence.

Smoothing

$$k=1,\ t=2$$

$$P(R_1=r_1 \mid U_1 \wedge U_2)$$

$$= \alpha\, \underbrace{P(R_1=r_1 \mid U_1)}_{\langle 0.818,\ 0.182 \rangle}\, P(U_2 \mid R_1=r_1)$$

Backward Recursion

$$P(U_2 \mid R_1=r_1)$$

$U_3 \wedge \cdots \wedge U_2$

$$= \sum_{r_2} P(U_2 \mid R_2=r_2)\, P(\ \mid r_2)\, P(R_2=r_2 \mid R_1=r_1)$$

$$= (\underbrace{0.9 * 1 * \langle 0.7,\ 0.3 \rangle}_{\langle 0.63,\ 0.27 \rangle}) + (\underbrace{0.2 * 1 * \langle 0.3,\ 0.7 \rangle}_{\langle 0.06,\ 0.14 \rangle})$$

$$= \langle 0.69,\ 0.41 \rangle$$

$$P(R_1=r_1 \mid U_1 \wedge U_2)$$

$$= \alpha\ \langle 0.818,\ 0.182 \rangle * \langle 0.69,\ 0.41 \rangle$$

$$\cong \langle 0.883,\ 0.117 \rangle$$

0.883

The smoothed estimate is higher than the filtered estimate 0.818. $U_2 = t$ makes it more likely to have rained on day 2, in turn, because rain tends to persist, that makes it more likely to haved rained on day 1.