Program Verification
While Loops

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Lecture 20

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Outline

Learning Goals

By the end of this lecture, you should be able to:

Partial correctness for while loops

- Determine whether a given formula is an invariant for a while loop.
- Find an invariant for a given while loop.
- Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops

- Determine whether a given formula is a variant for a while loop.
- Find a variant for a given while loop.
- Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.

Proving Total Correctness of While Loops

- Partial correctness
- Termination

Proving Partial Correctness of While Loops

$(\!| P |\!)$
$(\!| I |\!)$ implied (A)
while (B) {
  $(\!| (I \wedge B) |\!)$ partial-**while**
  C
  $(\!| I |\!)$ <justify based on C – a subproof>
}
$(\!| (I \wedge (\neg B)) |\!)$ partial-**while**
$(\!| Q |\!)$ implied (B)

Proof of implied (A): $(P \rightarrow I)$
Proof of implied (B): $((I \wedge (\neg B)) \rightarrow Q)$

I is called a loop invariant. We need to determine I!

What is a loop invariant?

A loop invariant is:

- A relationship among the variables. (A predicate formula involving the variables.)
- The word "invariant" means something that does not change.
- It is true before the loop begins.
- It is true at the start of every iteration of the loop and at the end of every iteration of the loop.
- It is true after the loop ends.

Proving partial correctness of while loops

Indicate the places in the program where the loop invariant is true.

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```

$(\!|(y = x!)|\!)$

Proving partial correctness of a while loop

Steps to follow:

- Find a loop invariant.
- Complete the annotations.
- Prove any implied's.

How do we find a loop invariant???

How do we find a loop invariant?

First, we need to understand the purpose of an invariant.

- The postcondition is the ultimate goal of our while loop.
- At every iteration, we are making progress towards the postcondition.
- The invariant is describing the progress we are making at every iteration.

Partial While - Example 1

Example 1:
Prove that the following triple is satisfied under partial correctness.

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```

$(\!|(y = x!)|\!)$

Finding a loop invariant

Step 1: Write down the values of all the variables every time the while test is reached.

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
    z = z + 1;
    y = y * z;
}
```

$(\!|(y = x!)|\!)$

Finding a loop invariant

Step 2: Find relationships among the variables that are true for every while test. These are our candidate invariants.

Come up with some invariants in the next 2 minutes.

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```

$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
5	4	$24 = 4!$
5	5	$120 = 5!$

CQ 1 Is this a loop invariant?

CQ 1: Is $(\neg(z = x))$ a loop invariant?

(A) Yes (B) No (C) I don't know...

$(\!|(x \geq 0)|\!)$
```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```
$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
5	4	$24 = 4!$
5	5	$120 = 5!$

CQ 2 Is this a loop invariant?

CQ 2: Is $(z \leq x)$ a loop invariant?

(A) Yes (B) No (C) I don't know...

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```

$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
5	4	$24 = 4!$
5	5	$120 = 5!$

CQ 3 Is this a loop invariant?

CQ 3: Is $(y = z!)$ a loop invariant?

(A) Yes (B) No (C) I don't know...

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```

$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
5	4	$24 = 4!$
5	5	$120 = 5!$

CQ 4 Is this a loop invariant?

CQ 4: Is $(y = x!)$ a loop invariant?

(A) Yes (B) No (C) I don't know...

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```

$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
5	4	$24 = 4!$
5	5	$120 = 5!$

CQ 5 Is this a loop invariant?

CQ 5: Is $((z \leq x) \wedge (y = z!))$ a loop invariant?

(A) Yes (B) No (C) I don't know...

$(\!|(x \geq 0)|\!)$
```
y = 1;
z = 0;
while (z != x) {
   z = z + 1;
   y = y * z;
}
```
$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
5	4	$24 = 4!$
5	5	$120 = 5!$

Finding a loop invariant

Step 3: Try each candidate invariant until we find one that works for our proof.

$(\!|(x \geq 0)|\!)$

```
y = 1;
z = 0;
while (z != x) {
    z = z + 1;
    y = y * z;
}
```

$(\!|(y = x!)|\!)$

x	z	y
5	0	$1 = 0!$
5	1	$1 = 1!$
5	2	$2 = 2!$
5	3	$6 = 3!$
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5	5	$120 = 5!$

How do we find an invariant?

A recap of the steps to find an invariant:

- Write down the values of all the variables every time the while test is reached.
- Find relationships among the variables that are true for every while test. These are our candidate invariants.
- Try each candidate invariant until we find one that works for our proof.

Partial While - Example 1 ($(z \leq x)$ as the invariant)

$(\!|(x \geq 0)|\!)$

$(\!|(0 \leq x)|\!)$ implied (A)

```
y = 1;
```

$(\!|(0 \leq x)|\!)$ assignment

```
z = 0;
```

$(\!|(z \leq x)|\!)$ assignment

```
while (z != x) {
```

  $(\!|((z \leq x) \wedge (\neg(z = x)))|\!)$ partial-**while**

  $(\!|(z+1 \leq x)|\!)$ implied (B)

```
  z = z + 1;
```

  $(\!|(z \leq x)|\!)$ assignment

```
  y = y * z;
```

  $(\!|(z \leq x)|\!)$ assignment

```
}
```

$(\!|((z \leq x) \wedge (\neg(\neg(z = x))))|\!)$ partial-**while**

$(\!|(y = x!)|\!)$ implied (C)

CQ 7 Is there a proof for implied (A)?

We used $(z \leq x)$ as the invariant.

CQ 7: Is there a proof for implied (A)?

$$((x \geq 0) \rightarrow (0 \leq x))$$

(A) Yes

(B) No

(C) I don't know.

CQ 8 Is there a proof for implied (B)?

We used $(z \leq x)$ as the invariant.

CQ 8: Is there a proof for implied (B)?

$$(((z \leq x) \wedge (\neg(z = x))) \rightarrow (z + 1 \leq x))$$

(A) Yes

(B) No

(C) I don't know.

CQ 9 Is there a proof for implied (C)?

We used $(z \leq x)$ as the invariant.

CQ 9: Is there a proof for implied (C)?

$$(((z \leq x) \wedge (\neg(\neg(z = x)))) \rightarrow (y = x!))$$

(A) Yes

(B) No

(C) I don't know.

Partial While - Example 2

Example 2:
Prove that the following triple is satisfied under partial correctness.

$(\!|(x \geq 0)|\!)$
```
y = 1;
z = 0;
while (z < x) {
   z = z + 1;
   y = y * z;
}
```
$(\!|(y = x!)|\!)$

Let's try using $(y = z!)$ as the invariant in our proof.

Which invariant leads to a valid proof?

To check whether an invariant leads to a valid proof,
we need to check whether all of the implied's can be proved.

CQ 11 Is there a proof for implied (A)?

We used $(y = z!)$ as the invariant.

CQ 11: Is there a proof for implied (A)?

$$((x \geq 0) \rightarrow (1 = 0!))$$

(A) Yes

(B) No

(C) I don't know.

CQ 12 Is there a proof for implied (B)?

We used $(y = z!)$ as the invariant.

CQ 12: Is there a proof for implied (B)?

$$(((y = z!) \wedge (z < x)) \rightarrow (y * (z + 1) = (z + 1)!))$$

(A) Yes

(B) No

(C) I don't know.

CQ 13 Is there a proof for implied (C)?

We used $(y = z!)$ as the invariant.

CQ 13: Is there a proof for implied (C)?

$$(((y = z!) \wedge (\neg(z < x))) \rightarrow (y = x!))$$

(A) Yes

(B) No

(C) I don't know.

CQ 14 Is there a proof for implied (C)?

We used $((y = z!) \wedge (z \leq x))$ as the invariant.

CQ 14: Is there a proof for implied (C)?

$$(((y = z!) \wedge (z \leq x)) \wedge (\neg(z < x))) \rightarrow (y = x!))$$

(A) Yes

(B) No

(C) I don't know.

Proving Termination

Find an integer expression that

- is non-negative before the loop starts, at every iteration of the loop, and after the loop ends.
- decreases by at least 1 at every iteration of the loop.

This integer expression is called a variant (something that changes).

The loop must terminate because a non-negative integer can decrease by 1 a finite number of times.

Example 2: Finding a variant

Example 2:
Prove that the following program terminates.

```
y = 1;
z = 0;
while (z < x) {
   z = z + 1;
   y = y * z;
}
```

How do we find a variant? The loop guard ($z < x$) helps.

Example 2: Proof of Termination

Consider the variant $(x - z)$.

Before the loop starts, $(x - z) \geq 0$ because the precondition is $(x \geq 0)$ and the second assignment mutates z to be 0.

During every iteration of the loop, $(x - z)$ decreases by 1 because x does not change and z increases by 1.

Thus, $x - z$ will eventually reach 0.

When $x - z = 0$, the loop guard $z < x$ will terminate the loop.

Revisiting the learning goals

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Total correctness for while loops

- Determine whether a given formula is a variant for a while loop.
- Find a variant for a given while loop.
- Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.