Soundness and Completeness of Natural Deduction   Oct 5

Let $\Sigma = \{P_1, P_2, \ldots, P_n\}$ be a set of propositional formulas.
Let c be a propositional formula.

We want to establish semantic entailment.

$\Sigma \models c$ if and only if:
entails.

For any truth valuation t, if all the premises in Σ are true under t ($\Sigma^t = T$), then the conclusion c is true under t. ($c^t = T$)

(This definition is equivalent to the following definition:
$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow c$ is a tautology.)

Several ways to establish/prove semantic entailment:
- truth table.
- direct proof: consider every valuation for which all the premises are true, show that the conclusion is true.)
- proof by contradiction.
- ☆ natural deduction.

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Natural deduction is a proof system in propositional logic.
There are other proof systems:

- resolution (1 inference rule)
- axiomatic systems
- semantic tableaux

A proof:

- starts with a set of premises Σ.
- transforms the premises using a set of rules.
- ends with the conclusion.

A proof is purely syntactic:

- Given the rules, we can check the correctness of the proof without understanding its meaning.
- In fact, a machine can do this check for us.

We write $\Sigma \vdash C$ or $\Sigma \;_{ND}\!\vdash C$ if and only if.

There exists a (natural deduction) proof that transforms the premises in Σ into the conclusion C.

You may have realized that

$$\underbrace{\Sigma \models C}_{\text{meaning and validity of an argument}} \neq \underbrace{\Sigma \vdash C}_{\text{mechanical manipulation of symbols}}$$

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Idealy, we want $\Sigma \models C$ and $\Sigma \vdash C$ to be equivalent.
This can mean two properties:

(Soundness).
If $\underbrace{\text{there is a proof from } \Sigma \text{ to } C}_{\Sigma \vdash C}$, then $\underbrace{\Sigma \text{ entails } C}_{\Sigma \models C}$

$$\Sigma \vdash C \longrightarrow \Sigma \models C.$$

(If I can prove something, then it's true.)
(Every formula I can prove in this system is sound.)

(Completeness).
If $\underbrace{\Sigma \text{ entails } C}_{\Sigma \models C}$, then $\underbrace{\text{I can construct a proof from } \Sigma \text{ to } C}_{\Sigma \vdash C}$.

$$\Sigma \models C \longrightarrow \Sigma \vdash C$$

(If something is true, then I can prove it.)
(I can prove every valid entailment in this system).

When we are using natural deduction as a proof system, we are taking soundness and completeness <u>for granted</u>.

Theorem: Natural deduction is both sound and complete.

Properties of other proof systems.

① Intuitionistic logic: sound but not complete.
  e.g. does not prove $(P \vee (\neg P))$.

② a system that is not sound but complete.
  e.g. add $P \wedge (\neg P)$ as an axiom.
  - not sound, because we can prove $P \wedge (\neg P)$, which is false.
  - complete, assume $P \wedge (\neg P)$ and we can derive anything.

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Examples of using soundness and completeness.

① Show that there does not exist a (natural deduction) proof for
$\{(P \vee Q)\} \vdash P$.

Proof:

The natural deduction proof system is sound,
so if $\{(P \vee Q)\} \vdash P$, then $\{(P \vee Q)\} \models P$.
Take the contrapositive, we have that
if $\{(P \vee Q)\} \not\models P$, then $\{(P \vee Q)\} \nvdash P$.
It is sufficient to show that the entailment does not hold.

(.... show the entailment does not hold.)

Therefore, there is no proof for $\{(P \vee Q)\} \vdash P$.

QED.

② True or False.

(a) if $\phi \nvdash C$, then $\phi \not\models C$, where C is any propositional formula.
True by the contrapositive of the completeness theorem.

(b) if $\{P_1, P_2\} \not\models C$, then $\phi \not\models ((P_1 \wedge P_2) \to C)$.
True by the definition of entailment.

(c) if $\phi \not\models ((P_1 \wedge P_2) \to C)$, then $\{P_1, P_2\} \not\models C$.
True by the definition of entailment.

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Theorem (soundness of natural deduction):
if $\Sigma \vdash C$ is valid, then $\Sigma \vDash C$ holds.

We prove this by structural induction on the proof for $\Sigma \vdash C$.

A proof is a recursive structure.
A proof either

Base case ① does not use any inference rule to derive the conclusion, or

Induction step ② uses an inference rule on one or more (sub) proofs to derive the conclusion.

Proof of the soundness theorem:
We prove the theorem by structural induction on the proof for $\Sigma \vdash C$.

Base case: C is a premise.
If $C \in \Sigma$, and $\Sigma^t = T$ for some valuation t, then $C^t = T$ and $\Sigma \vDash C$.

Induction step:
Consider several cases for the last rule applied in the proof.

case ①: The rule is $\wedge i$ with premises $\Sigma \vdash a$ and $\Sigma \vdash b$, and reached the conclusion $(a \wedge b)$.
Induction hypothesis: $\Sigma \vDash a$ and $\Sigma \vDash b$.
We need to prove that $\Sigma \vDash (a \wedge b)$.
Consider a valuation such that $\Sigma^t = T$.
Since $\Sigma \vDash a$, $a^t = T$. Since $\Sigma \vDash b$, $b^t = T$.
Thus, $(a \wedge b)^t = T$ and $\Sigma \vDash (a \wedge b)$.

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Proof of the soundness theorem continued:

case ② The rule is $\to e$ with premises $\Sigma \vdash a$ and $\Sigma \vdash (a \to c)$.
Induction hypothesis: $\Sigma \models a$ and $\Sigma \models (a \to c)$
We need to prove that $\Sigma \models c$.
Consider a valuation t such that $\Sigma^t = T$.
Since $\Sigma \models a$, $a^t = T$.
Since $\Sigma \models (a \to c)$, $(a \to c)^t = T$.
$a^t = T$ and $(a \to c)^t = T$, so $c^t = T$.
Therefore, $\Sigma \models c$.

.... (omitting many cases here) ...

By the principle of structural induction,
if $\Sigma \vdash c$ is valid, then $\Sigma \models c$ holds

QED

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Theorem (completeness of natural deduction)
if $\Sigma \models C$ holds, then $\Sigma \vdash C$ is valid.
(let $\Sigma = \{P_1, P_2, \ldots, P_n\}$.)

Proof sketch:
step 1: show that $\emptyset \models P_1 \to (P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots)))$ holds.
step 2: show that $\emptyset \vdash P_1 \to (P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots)))$ is valid.
step 3: show that $\{P_1, \ldots, P_n\} \vdash C$ is valid.

step 1: if $\{P_1, P_2, \ldots, P_n\} \models C$ holds,
then $\models P_1 \to (P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots)))$ holds.

(We can prove this by a direct proof or a proof by contradiction.)

step 3: given the proof $\vdash P_1 \to (P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots)))$,
we construct a proof for $\{P_1, P_2, \ldots, P_n\} \vdash C$.

1.	$P_1 \to (P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots)))$	
2.	P_1	premise
3.	$P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots))$	$\to e$: 1, 2
4.	P_2	premise
5.	$P_3 \to (\cdots (P_n \to C) \cdots))$	$\to e$: 3, 4
	$\vdots$	
	C	$\to e$:

(Introduce $P_1, P_2, \ldots, P_n$ as premises.
Apply $\to e$ n times to get to C.)

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Proof of the completeness theorem continued:

step 2: we need to construct a proof for

$$\vdash \underbrace{P_1 \to (P_2 \to (P_3 \to (\cdots (P_n \to C) \cdots)))}_{\varphi}$$

For each line of φ's truth table, we can construct a proof for it.

Example: $\{(\neg q), (P \to q)\} \models (\neg P)$.

we need to construct a proof for $\vdash \underbrace{((\neg q) \to ((P \to q) \to (\neg P)))}_{\varphi}$.

P	q	φ		
0	0	1	$\{(\neg P), (\neg q)\} \vdash \varphi$	For now, assume
0	1	1	$\{(\neg P), q\} \vdash \varphi$.	we can construct
1	0	1	$\{P, (\neg q)\} \vdash \varphi$.	these proofs
1	1	1	$\{P, q\} \vdash \varphi$.	(lemma)

The proof for $\vdash ((\neg q) \to ((P \to q) \to (\neg P)))$

Proof: $P \vee (\neg P)$ law of excluded middle. (LEM)

P assumption	$(\neg P)$ assumption.
$q \vee (\neg q)$ LEM	$q \vee (\neg q)$ LEM
q assumption / $\vdots$ / φ ‖ $(\neg q)$ assumption / $\vdots$ / φ	q assumption / $\vdots$ / φ ‖ $(\neg q)$ assumption / $\vdots$ / φ
φ $\vee e$	φ $\vee e$

φ $\vee e$.

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Proof of the completeness theorem continued:

Step 2: Lemma: Consider a formula φ which contains propositional variables $P_1, P_2, \ldots, P_n$.
Define $\hat{P_1}, \hat{P_2}, \ldots, \hat{P_n}$ below. (for each line of φ's truth table)
if P_i is true in this line, $\hat{P_i} = P_i$.
if P_i is false in this line, $\hat{P_i} = (\neg P_i)$.

For each line of φ's truth table, we can construct a proof for

$\{\hat{P_1}, \hat{P_2}, \hat{P_3}, \ldots, \hat{P_n}\} \vdash \varphi$ if φ is true.
$\{\hat{P_1}, \hat{P_2}, \hat{P_3}, \ldots, \hat{P_n}\} \vdash (\neg\varphi)$ if φ is false.

Example:

$\overbrace{P}^{P_1}$	$\overbrace{q}^{P_2}$	$\overbrace{(P \land q)}^{\varphi}$	$\overbrace{\quad}^{\hat{P_1}}$ $\overbrace{\quad}^{\hat{P_2}}$ $\overbrace{\quad}^{\varphi}$
0	0	0	$\{(\neg P), (\neg q)\} \vdash (P \land q)$
0	1	0	$\{(\neg P), q\} \vdash (P \land q)$
1	0	0	$\{P, (\neg q)\} \vdash (P \land q)$
1	1	1	$\{P, q\} \vdash (P \land q)$.

Proof of lemma by structural induction on φ.
base case: φ is a propositional variable.
induction step:
case 1: $\varphi = (\neg \chi)$.
Induction hypothesis: For each line of χ's truth table, there is a proof for
$\{\hat{P_1}, \hat{P_2}, \ldots, \hat{P_n}\} \vdash \chi$ if χ is true.
$\{\hat{P_1}, \hat{P_2}, \ldots, \hat{P_n}\} \vdash (\neg\chi)$ if χ is false.
We need to prove that
$\{\hat{P_1}, \hat{P_2}, \ldots, \hat{P_n}\} \vdash \varphi$ if φ is true.
$\{\hat{P_1}, \hat{P_2}, \ldots, \hat{P_n}\} \vdash (\neg\varphi)$ if φ is false.