

Natural Deduction Examples

$$\textcircled{1} \{ (\forall x (P(x) \rightarrow Q(x))) \} \vdash ((\forall x P(x)) \rightarrow (\forall y Q(y)))$$

Proof:	1	$(\forall x (P(x) \rightarrow Q(x)))$	premise.
	2	$(\forall x P(x))$	assumption
	3	$P(w)$	$\forall e: 2$
	4	$P(w) \rightarrow Q(w)$	$\forall e: 1$
	5	$Q(w)$	$\rightarrow e: 4, 3$
	6	u fresh NOW WHAT?	
	7		
	8	$Q(u)$	
	9	$(\forall y Q(y))$	$\forall i: 6-8$
	10	$((\forall x P(x)) \rightarrow (\forall y Q(y)))$	$\rightarrow i: 2-9$

In the box 2-9,

- I used $\forall e$ on 1 and 2 to derive $Q(w)$ for some w . This appears to be good progress because I eventually want to show that $Q(y)$ is true for all y .
- However, to get to $(\forall y Q(y))$, we need to use $\forall i$, which requires us to start a subproof (6-8) and select a fresh variable u in this subproof. Any formula using u cannot appear outside of box 6-8. This means we cannot replace w by u and use $Q(u)$ in box 6-8.
- We could copy $Q(w)$ into box 6-8 using reflexivity, but knowing $Q(w)$ doesn't help us to conclude $Q(u)$. u and w are likely different.
- Therefore, we have to use $\forall e$ on 1 and 2 again in the box 6-8, where we can substitute u into the formula.

Natural Deduction Examples.

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$$\textcircled{2} \{ (\exists y (\forall x \neg P(x, y))) \} \vdash (\forall x (\exists y P(x, y)))$$

Proof:	1	$(\exists y (\forall x P(x, y)))$	premise
	2	$(\forall x P(x, w))$, w fresh	assumption
	3	$P(w, w)$	$\forall e: 2$
	4	$P(t, w)$	$\forall e: 2$
	5	s fresh	
	6		
	7	$P(s, r)$	$\forall e: 2$
	8	$(\exists y P(s, y))$	$\exists i: 7$
	9	$(\forall x (\exists y P(x, y)))$	$\forall i: 5-8$
	10	$(\forall x (\exists y P(x, y)))$	$\exists e: 1, 2-9$

These steps are NOT useful for this proof

- On lines 3 and 4, I tried using $\forall e$ on line 2 using the variable w and the variable t . However, neither $P(w, w)$ nor $P(t, w)$ help me to conclude $(\forall x (\exists y P(x, y)))$.
- Instead, I need to start a subproof 5-8 and choose a fresh variable s . Any formula containing s cannot escape the box 5-8.
- We could copy $P(w, w)$ and $P(t, w)$ into box 5-8, but they don't help us to conclude $(\exists y P(s, y))$. We need to apply $\forall e$ inside the box 5-8 in order to conclude $(\exists y P(s, y))$.