Predicate Logic: Soundness and Completeness of Formal Deduction

Alice Gao

Lecture 17

Outline

Learning goals

By the end of this lecture, you should be able to:

- Define soundness and completeness.
- Prove that an inference rule is sound or not sound.
- Prove that a logical consequence holds using the soundness and completeness theorems.
- Show that no natural deduction proof exists for a logical consequence using the soundness and completeness theorems.

The Soundness of Formal Deduction

Theorem

Formal Deduction for Predicate Logic is sound.

Proof Sketch.

Since Formal Deduction for Propositional Logic is sound, it suffices to prove the soundness of the $\forall-$, $\exists+$, $\forall+$ and $\exists-$ rules. □

The soundness of $\forall-$

Theorem

The $\forall-$ inference rule is sound.
That is, if $\Sigma \vDash \forall x\ A(x)$, then $\Sigma \vDash A(t)$ where t is a term.

The soundness of $\exists+$

Theorem

The $\exists+$ inference rule is sound.
That is, if $\Sigma \vDash A(t)$, then $\Sigma \vDash \exists x\ A(x)$ where t is a term.

The soundness of $\exists-$

Theorem

The $\exists-$ inference rule is sound.
That is, if $\Sigma, A(u) \vDash B$, u not occurring in Σ or B,
then $\Sigma, \exists x\ A(x) \vDash B$.

The soundness of $\forall+$

Theorem

The $\forall+$ inference rule is sound.
That is, if $\Sigma \models A(u)$, u not occurring in Σ, then $\Sigma \models \forall x\ A(x)$.

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define soundness and completeness.
- Prove that an inference rule is sound or not sound.
- Prove that a semantic entailment holds using the soundness and completeness theorems.
- Show that no natural deduction proof exists for a semantic entailment using the soundness and completeness theorems.