Predicate Logic: Formal Deduction

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Lecture 16

Outline

Learning goals

By the end of this lecture, you should be able to:

- Describe the rules of inference for formal deduction for predicate logic.
- Prove that a conclusion follows from a set of premises using formal deduction inference rules.

Outline

Forall-elimination

$\forall$-elimination ($\forall-$)

$$\begin{aligned} \text{if } & \Sigma \vdash \forall x\, A(x), \\ \text{then } & \Sigma \vdash A(t). \end{aligned}$$

Compare this to $\wedge$-elimination ($\wedge-$)

$$\begin{aligned} \text{if } & \Sigma \vdash A \wedge B, \\ \text{then } & \Sigma \vdash A. \\ \text{if } & \Sigma \vdash A \wedge B, \\ \text{then } & \Sigma \vdash B. \end{aligned}$$

Exercise: Forall-elimination

Show that

$$P(u), \forall x(P(x) \to \neg Q(x)) \vdash \neg Q(u).$$

Outline

Exists-introduction

$\exists$-introduction ($\exists+$)

$$\text{if } \Sigma \vdash A(t),$$
$$\text{then } \Sigma \vdash \exists x\, A(x).$$

where $A(x)$ results by replacing some (not necessarily all) occurrences of t in $A(t)$ by x.

Compare this to $\vee$-introduction ($\vee+$)

$$\text{if } \Sigma \vdash A,$$
$$\text{then } \Sigma \vdash A \vee B.$$
$$\text{if } \Sigma \vdash B,$$
$$\text{then } \Sigma \vdash A \vee B.$$

CQ Exists-introduction

Proof 1:

$$\begin{array}{lll} (1) & \Sigma \vdash (P(v) \to Q(v)) & \text{by assumption} \\ (2) & \Sigma \vdash (\exists x\ (P(x) \to Q(v))) & \text{by } (\exists+, 1) \end{array}$$

Proof 2:

$$\begin{array}{lll} (1) & \Sigma \vdash (P(v) \to Q(v)) & \text{by assumption} \\ (2) & \Sigma \vdash (\exists x\ (P(x) \to Q(x))) & \text{by } (\exists+, 1) \end{array}$$

Which of the following is a correct application of the $\exists+$ rule?

(A) Both proofs

(B) Proof 1 only

(C) Proof 2 only

(D) Neither proof

Exercise: Exists-introduction

Show that

$$\{(\neg P(v))\} \vdash (\exists x\ (P(x) \to Q(v))).$$

Outline

Forall-introduction

$\forall$-introduction ($\forall+$)

$$\begin{aligned} &\text{if } \Sigma \vdash A(u),\ u \text{ not occurring in } \Sigma, \\ &\text{then } \Sigma \vdash \forall x\, A(x). \end{aligned}$$

Compare this to $\wedge$-introduction ($\wedge+$)

$$\begin{aligned} \text{if } &\Sigma \vdash A, \\ &\Sigma \vdash B, \\ \text{then } &\Sigma \vdash A \wedge B. \end{aligned}$$

Exercise: Forall-introduction

Show that

$$(\forall x\ (P(x) \rightarrow Q(x))) \vdash ((\forall x\ P(x)) \rightarrow (\forall y\ Q(y))).$$

Outline

Exists-elimination

$\exists$-elimination ($\exists-$)

$$\begin{aligned} &\text{if } \Sigma, A(u) \vdash B, u \text{ not occurring in } \Sigma \text{ or } B, \\ &\text{then } \Sigma, \exists x\, A(x) \vdash B. \end{aligned}$$

Compare this to $\vee$-elimination ($\vee-$)

$$\begin{aligned} \text{if } \Sigma, A &\vdash C, \\ \Sigma, B &\vdash C, \\ \text{then } \Sigma, A \vee B &\vdash C. \end{aligned}$$

Exercise: Exists-elimination

Show that

$$\exists x\ (P(x) \vee Q(x)) \vdash (\exists x\ P(x) \vee (\exists x\ Q(x)).$$

Outline

Putting them together

Show that

$$\exists y\ (\forall x\ P(x,y)) \vdash \forall x\ (\exists y\ P(x,y)).$$

Which rule should we apply next?

(A) $\forall+$

(B) $\forall-$

(C) $\exists+$

(D) $\exists-$

(E) Another rule

Revisiting the learning goals

By the end of this lecture, you should be able to:

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