Propositional Logic: Completeness of Formal Deduction

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Lecture 10

Learning Goals

By the end of this lecture, you should be able to

- Define the completeness of formal deduction.
- Define consistency and satisfiability.
- Prove properties of consistent and satisfiable sets based on their definitions.
- Reproduce the key steps of the proof of the completeness theorem.

The Soundness of Formal Deduction

Theorems 1 and 2 are equivalent.

Theorem 1 (Soundness of Formal Deduction)
If $\Sigma \vdash A$*, then* $\Sigma \models A$*.*

Theorem 2
If Σ *is satisfiable, then* Σ *is consistent.*

The Completeness of Formal Deduction

Theorems 3 and 4 are equivalent.

Theorem 3 (Completeness of Formal Deduction)

If $\Sigma \models A$, *then* $\Sigma \vdash A$.

Theorem 4

If Σ *is consistent, then* Σ *is satisfiable.*

Outline

Σ is satisfiable

Definition 5

Σ is satisfiable if there exists a truth valuation t such that for every $A \in \Sigma$, $A^t = 1$.

Note that this is a semantic notion.

Σ is not satisfiable.
For every truth valuation t, there exists $A \in \Sigma$, $A^t = 0$.

Σ is consistent

Intuitively, Σ is consistent if it doesn't prove a contradiction.

Two equivalent definitions:

1. There exists a formula A, $\Sigma \nvdash A$.
   $\exists A\, (\Sigma \nvdash A)$.
   Negating the definition: For every formula A, $\Sigma \vdash A$.
   $\forall A\, (\Sigma \vdash A)$.
2. For every formula A, if $\Sigma \vdash A$, then $\Sigma \nvdash (\neg A)$.
   $\forall A\, (\Sigma \vdash A \rightarrow \Sigma \nvdash \neg A)$.
   Negating the definition: There exists a formula A, $\Sigma \vdash A$ and $\Sigma \vdash (\neg A)$. $\exists A\, (\Sigma \vdash A \wedge \Sigma \vdash \neg A)$.

Note that consistency is a syntactical notion.
Let's prove that these two definitions are equivalent.

Σ is consistent - two equivalent definitions

Theorem 6

Def 2 implies def 1.

Proof.

Assume that for every formula A, if $\Sigma \vdash A$, then $\Sigma \nvdash (\neg A)$.
We need to find a formula A such that $\Sigma \nvdash A$.

Consider any propositional formula A.

- If $\Sigma \nvdash A$, we are done.
- If $\Sigma \vdash A$, then by our assumption, it must be that $\Sigma \nvdash (\neg A)$. Therefore, $(\neg A)$ is the formula that we need and we are done.

□

Σ is consistent - two equivalent definitions

Theorem 7

Negation of def 2 implies negation of def 1.

Proof.

Assume that there exists a formula A such that $\Sigma \vdash A$ and $\Sigma \vdash (\neg A)$.

We need to prove that for every formula A, $\Sigma \vdash A$.

Consider any formula B.

(1)	$\Sigma \vdash A$	(by assumption)
(2)	$\Sigma \vdash (\neg A)$	(by assumption)
(3)	$\Sigma, (\neg B) \vdash A$	(by $(+), (1)$)
(4)	$\Sigma, (\neg B) \vdash (\neg A)$	(by $(+), (2)$)
(5)	$\Sigma \vdash B$	(by $(\neg -), (3), (4)$)

This proof shows that $\Sigma \vdash B$ for every formula B.

□

Sketch of the Proof of The Completeness of Formal Deduction

Theorem 8

If Σ is consistent implies Σ is satisfiable,
then $\Sigma \models A$ implies $\Sigma \vdash A$.

Proof Sketch.

Assume that $\Sigma \models A$.

If $\Sigma \models A$, then we can prove that $\Sigma \cup \{\neg A\}$ is not satisfiable.
(Part of assignment 4)

By our assumption, if $\Sigma \cup \{\neg A\}$ is not satisfiable,
then $\Sigma \cup \{\neg A\}$ is inconsistent.

If $\Sigma \cup \{\neg A\}$ is inconsistent, then $\Sigma \vdash A$.
(Let's prove this part.)

□

Properties of a Consistent Set — Direction 1

Theorem 9

If $\Sigma \cup \{\neg A\}$ *is inconsistent, then* $\Sigma \vdash A$*.*

Proof.

Assume that $\Sigma \cup \{\neg A\}$ is inconsistent. Then, there exists a formula B such that $\Sigma \cup \{\neg A\} \vdash B$ and $\Sigma \cup \{\neg A\} \vdash \neg B$. We need to show that $\Sigma \vdash A$.

(1)	$\Sigma, (\neg A) \vdash B$	(by assumption)
(2)	$\Sigma, (\neg A) \vdash (\neg B)$	(by assumption)
(3)	$\Sigma \vdash A$	(by $(\neg -), (1), (2)$)

□

Similarly, we can prove that
"if $\Sigma \cup \{A\}$ is inconsistent, then $\Sigma \vdash (\neg A)$."

Exercise: Properties of a Consistent Set — Direction 2

Theorem 10

If $\Sigma \vdash A$, *then* $\Sigma \cup \{\neg A\}$ *is inconsistent.*

Proof.

Consider any formula B.

(1)	$\Sigma \vdash A$	(by assumption)
(2)	$\Sigma, (\neg A), (\neg B) \vdash A$	(by $(+), (1)$)
(3)	$\Sigma, (\neg A), (\neg B) \vdash \neg A$	(by $(\in)$)
(4)	$\Sigma, (\neg A) \vdash B$	(by $(\neg -), (2), (3)$)

□

Similarly, we can prove that
"if $\Sigma \vdash (\neg A)$, then $\Sigma \cup \{A\}$ is inconsistent."

Outline

Two Proofs of the Completeness of Formal Deduction

We will present two versions of the proofs of the completeness of formal deduction.
These two versions are almost identical except for two key points.

1. The proofs define the truth valuation t based on the maximally consistent set Σ^*.
   - Proof 1 defines $p^t = 1$ iff $p \in \Sigma^*$.
   - Proof 2 defines $p^t = 1$ iff $\Sigma^* \vdash p$.
2. Because of the definitions of the truth valuation t, the proofs require different definitions of maximal consistency.
   - Proof 1 requires the maximally consistent set Σ^* to satisfy $A \in \Sigma^*$ or $(\neg A) \in \Sigma^*$ for every formula A.
   - Proof 2 requires the maximally consistent set Σ^* to satisfy $\Sigma^* \vdash A$ or $\Sigma^* \vdash (\neg A)$ for every formula A.

Two Definitions of Maximal Consistency

The two proofs require two different definitions of a maximally consistent set. The first definition is stronger than and implies the second definition.

1. Stronger definition given in the Lu Zhongwan textbook

   Given a consistent Σ, Σ is maximally consistent if and only if
   - For every formula A, if $A \notin \Sigma$, then $\Sigma \cup \{A\}$ is inconsistent.
   - For every formula A, $A \in \Sigma$ or $(\neg A) \in \Sigma$ but not both.

   This definition is re-stated on slide 18.

2. Weaker definition given in Assignment 5

   Given a consistent Σ, Σ is maximally consistent if and only if
   - For every formula A, if $\Sigma \nvdash A$, then $\Sigma \cup \{A\}$ is inconsistent.
   - For every formula A, $\Sigma \vdash A$ or $\Sigma \vdash (\neg A)$ but not both.

   This definition is re-stated on slide 29.

Outline

Every Consistent Set is Satisfiable

To finish the proof of the completeness theorem,
it remains to prove theorem 4,
which says "if Σ is consistent, then Σ satisfiable."

Proof Sketch.
Assume that Σ is consistent. We need to find a truth valuation t such that $A^t = 1$ for every formula $A \in \Sigma$.

Extend Σ to some maximally consistent set Σ^*. Let t be a truth valuation such that for every propositional variable p,
$p^t = 1$ if and only if $p \in \Sigma^*$.

For every $A \in \Sigma$, $A \in \Sigma^*$. We can prove that $A^t = 1$. Therefore, Σ is satisfied by t. □

Definitions of a Maximally Consistent Set (Stronger Version)

A key step in proving theorem 4 is
to construct a maximally consistent set that includes Σ.

First, let's look at the definition of a maximally consistent set.

Given a consistent Σ, Σ is maximally consistent if and only if

- For every formula A, if $A \notin \Sigma$, then $\Sigma \cup \{A\}$ is inconsistent.
- For every formula A, $A \in \Sigma$ or $(\neg A) \in \Sigma$ but not both.

This definition is given in the Lu Zhongwan textbook and
it is stronger than the definition on slide 29.

Extending Σ to a Maximally Consistent Set Σ^*

Let Σ be a consistent set of formulas.
We extend Σ to a maximally consistent set Σ^* as follows.

Arbitrarily enumerate all the well-formed formulas using the following sequence.

$$A_1, A_2, A_3, \ldots$$

Construct an infinite sequence of sets Σ_n as follows.

$$\begin{cases} \Sigma_0 = \Sigma \\ \Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_{n+1}\}, \text{ if } \Sigma_n \cup \{A_{n+1}\} \text{ is consistent} \\ \Sigma_n, \text{ otherwise} \end{cases} \end{cases}$$

Observe that $\Sigma_n \subseteq \Sigma_{n+1}$ and Σ_n is consistent.
(We can prove this by induction on n.)

Extending to Maximal Consistency (continued)

Define $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma_n$.

Think of Σ^* as the largest possible set that

- contains Σ, and
- is consistent.

We will now prove that Σ^* is maximally consistent.

Σ^* is maximally consistent

First, we prove that Σ^* is consistent. Suppose that Σ^* is inconsistent. We need to derive a contradiction.
There is a finite subset $\{B_1, \ldots, B_k\} \subseteq \Sigma^*$, which is inconsistent. Suppose that $B_1 \in \Sigma_{i_1}, \ldots, B_k \in \Sigma_{i_k}$ and $i = \max(i_1, \ldots, i_k)$. By the construction of Σ_n, we have $\{B_1, \ldots, B_k\} \subseteq \Sigma_i$. Then, Σ_i is inconsistent, contradicting the construction of Σ_i. Hence, Σ^* is consistent.

Next, we prove that Σ^* is maximally consistent. Assume that B is a well-formed formula and $B \notin \Sigma^*$. We need to prove that $\Sigma^* \cup \{B\}$ is inconsistent.
B must be in the sequence $A_1, A_2, \ldots$. Let $B = A_{m+1}$. By the construction of Σ_n, since $B \notin \Sigma^*$, $\Sigma_m \cup \{A_{m+1}\}$ (or $\Sigma_m \cup \{B\}$) is inconsistent. Then, $\Sigma^* \cup \{B\}$ is inconsistent because $\Sigma_m \subseteq \Sigma^*$. Therefore, Σ^* is maximally consistent.

A Maximally Consistent Set Proves Its Elements

Note that direction 2 of this lemma does not hold for the weaker definitions of maximal consistency given in assignment 5.

Lemma 11 (Lemma 5.3.2 in Lu Zhongwan)

Suppose Σ *is maximally consistent. Then,* $A \in \Sigma$ *iff* $\Sigma \vdash A$*.*

Proof.

Direction 1: Assume $A \in \Sigma$. Then, $\Sigma \vdash A$ by $(\in)$.
Direction 2: Assume $\Sigma \vdash A$. Towards a contradiction, assume that $A \notin \Sigma$. Since Σ is maximally consistent, $\Sigma \cup \{A\}$ is inconsistent. Then, $\Sigma \vdash (\neg A)$ and Σ is inconsistent, contradicting the maximal consistency of Σ. Hence, $A \in \Sigma$. □

Satisfying a Maximally Consistent Set

Lemma 12

Let Σ^* be a maximally consistent set.
Let t be a truth valuation such that
$p^t = 1$ if and only if $p \in \Sigma^*$ for every propositional variable p.

Then, for every well-formed propositional formula A,
$A^t = 1$ if and only if $A \in \Sigma^*$.

Proof.

By induction on the structure of A.
(Continued..)

□

Base case and Inductive case 1

- Base case: A is a propositional variable p.
  $p \in \Sigma^*$ iff $p^t = 1$ by the definition of t.

- Inductive case 1: $A = \neg B$.
  Induction hypothesis: $B^t = 1$ iff $B \in \Sigma^*$.
  We need to show that $(\neg B)^t = 1$ iff $\neg B \in \Sigma^*$.

  $(\neg B)^t = 1$ iff $B^t = 0$ by the truth table of $\neg$.
  $B^t = 0$ iff $B \notin \Sigma^*$ by the induction hypothesis.
  $B \notin \Sigma^*$ iff $\neg B \in \Sigma^*$ because Σ^* is maximally consistent.
  Thus, $(\neg B)^t = 1$ iff $\neg B \in \Sigma^*$.

Inductive case 2

- Inductive case 2: $A = B \wedge C$.
  Induction hypotheses: $B^t = 1$ iff $B \in \Sigma^*$. $C^t = 1$ iff $C \in \Sigma^*$.
  We need to show that $(B \wedge C)^t = 1$ iff $B \wedge C \in \Sigma^*$.
  Direction 1:
  Assume $(B \wedge C)^t = 1$.
  Then, $B^t = 1$ and $C^t = 1$ by the truth table of $\wedge$.
  By the induction hypothesis, $B \in \Sigma^*$ and $C \in \Sigma^*$.
  By Lemma 11, $\Sigma^* \vdash B$ and $\Sigma^* \vdash C$.
  By $(\wedge+)$, $\Sigma^* \vdash B \wedge C$.
  By Lemma 11, $B \wedge C \in \Sigma^*$.

  Direction 2:
  Assume $B \wedge C \in \Sigma^*$.
  By Lemma 11, $\Sigma^* \vdash B \wedge C$.
  By $(\wedge-)$, $\Sigma^* \vdash B$ and $\Sigma^* \vdash C$.
  By Lemma 11, $B \in \Sigma^*$ and $C \in \Sigma^*$.
  By the induction hypothesis, $B^t = 1$ and $C^t = 1$.
  By the truth table of $\wedge$, $(B \wedge C)^t = 1$.

Inductive cases 3, 4, and 5

- Inductive case 3: $A = B \vee C$.
  Induction hypotheses: $B^t = 1$ iff $B \in \Sigma^*$. $C^t = 1$ iff $C \in \Sigma^*$.
  We can show that if $B \vee C \in \Sigma^*$ iff $B \in \Sigma^*$ or $C \in \Sigma^*$.

- Inductive case 4: $A = B \to C$.
  Induction hypotheses: $B^t = 1$ iff $B \in \Sigma^*$. $C^t = 1$ iff $C \in \Sigma^*$.
  We can show that $B \to C \in \Sigma^*$ iff $B \in \Sigma^*$ implies $C \in \Sigma^*$.

- Inductive case 5: $A = B \leftrightarrow C$.
  Induction hypotheses: $B^t = 1$ iff $B \in \Sigma^*$. $C^t = 1$ iff $C \in \Sigma^*$.
  We can show that $B \leftrightarrow C \in \Sigma^*$ iff $(B \in \Sigma^*$ iff $C \in \Sigma^*)$.

Outline

Every Consistent Set is Satisfiable

To finish the proof of the completeness theorem,
it remains to prove theorem 4,
which says "if Σ is consistent, then Σ satisfiable."

Proof Sketch.
Assume that Σ is consistent. We need to find a truth valuation t such that $A^t = 1$ for every formula $A \in \Sigma$.

Extend Σ to some maximally consistent set Σ^*. Let t be a truth valuation such that for every propositional variable p,
$p^t = 1$ if and only if $\Sigma^* \vdash p$.

For every $A \in \Sigma$, $A \in \Sigma^*$. We can prove that $A^t = 1$. Therefore, Σ is satisfied by t. □

Definitions of a Maximally Consistent Set (Weaker Version)

A key step in proving theorem 4 is
to construct a maximally consistent set that includes Σ.

Let's look at the definition of a maximally consistent set.

Given a consistent Σ, Σ is maximally consistent if and only if

- For every formula A, if $\Sigma \nvdash A$, then $\Sigma \cup \{A\}$ is inconsistent.
- For every formula A, $\Sigma \vdash A$ or $\Sigma \vdash (\neg A)$ but not both.

This definition is given in Assignment 5 and
it is weaker than the definition on slide 18.

Extending Σ to a Maximally Consistent Set Σ^*

Let Σ be a consistent set of formulas.
We extend Σ to a maximally consistent set Σ^* as follows.

Arbitrarily enumerate all the well-formed formulas using the following sequence.

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Construct an infinite sequence of sets Σ_n as follows.

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Observe that $\Sigma_n \subseteq \Sigma_{n+1}$ and Σ_n is consistent.
(We can prove this by induction on n.)

Extending to Maximal Consistency (continued)

Define $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma_n$.

Think of Σ^* as the largest possible set that

- contains Σ, and
- is consistent.

We will now prove that Σ^* is maximally consistent.

Σ^* is maximally consistent

First, we prove that Σ^* is consistent. Suppose that Σ^* is inconsistent. We need to derive a contradiction.
There is a finite subset $\{B_1, \ldots, B_k\} \subseteq \Sigma^*$, which is inconsistent.
Suppose that $B_1 \in \Sigma_{i_1}, \ldots, B_k \in \Sigma_{i_k}$ and $i = \max(i_1, \ldots, i_k)$. By the construction of Σ_n, we have $\{B_1, \ldots, B_k\} \subseteq \Sigma_i$. Then, Σ_i is inconsistent, contradicting the construction of Σ_i. Hence, Σ^* is consistent.

Next, we prove that Σ^* is maximally consistent.
Assume that B is a well-formed formula and $\Sigma^* \nvdash B$.
We need to prove that $\Sigma^* \cup \{B\}$ is inconsistent.
B must be in the sequence $A_1, A_2, \ldots$. Let $B = A_{m+1}$.
Since $\Sigma^* \nvdash B$, then $B \notin \Sigma^*$. By the construction of Σ_n, since $B \notin \Sigma^*$, $\Sigma_m \cup \{A_{m+1}\}$ (or $\Sigma_m \cup \{B\}$) is inconsistent. Then, $\Sigma^* \cup \{B\}$ is inconsistent because $\Sigma_m \subseteq \Sigma^*$. Therefore, Σ^* is maximally consistent.

Satisfying a Maximally Consistent Set

Lemma 13

Let Σ^* be a maximally consistent set.
Let t be a truth valuation such that
$p^t = 1$ if and only if $\Sigma^* \vdash p$ for every propositional variable p.

Then, for every well-formed propositional formula A,
$A^t = 1$ if and only if $\Sigma^* \vdash A$.

Proof.

By induction on the structure of A.
(Continued..)

□

Base case and Inductive case 1

- Base case: A is a propositional variable p.
  $\Sigma^* \vdash p$ iff $p^t = 1$ by the definition of t.

- Inductive case 1: $A = \neg B$.
  Induction hypothesis: $B^t = 1$ iff $\Sigma^* \vdash B$.
  We need to show that $(\neg B)^t = 1$ iff $\Sigma^* \vdash (\neg B)$.

  By the truth table of $\neg$, $(\neg B)^t = 1$ iff $B^t = 0$.
  By the induction hypothesis, $B^t = 0$ iff $\Sigma^* \nvdash B$.
  Since Σ^* is maximally consistent, $\Sigma^* \nvdash B$ iff $\Sigma^* \vdash (\neg B)$.

Inductive case 2

- Inductive case 2: $A = B \wedge C$.
  Induction hypotheses: $B^t = 1$ iff $\Sigma^* \vdash B$. $C^t = 1$ iff $\Sigma^* \vdash C$.
  We need to show that $(B \wedge C)^t = 1$ iff $\Sigma^* \vdash B \wedge C$.
  Direction 1:
  Assume $(B \wedge C)^t = 1$.
  By the truth table of $\wedge$, $B^t = 1$ and $C^t = 1$.
  By the induction hypothesis, $\Sigma^* \vdash B$ and $\Sigma^* \vdash C$.
  By $(\wedge+)$, $\Sigma^* \vdash B \wedge C$.

  Direction 2:
  Assume $\Sigma^* \vdash B \wedge C$.
  By $(\wedge-)$, $\Sigma^* \vdash B$ and $\Sigma^* \vdash C$.
  By the induction hypothesis, $B^t = 1$ and $C^t = 1$.
  By the truth table of $\wedge$, $(B \wedge C)^t = 1$.

Inductive cases 3, 4, and 5

- Inductive case 3: $A = B \vee C$.
  Induction hypotheses: $B^t = 1$ iff $\Sigma^* \vdash B$. $C^t = 1$ iff $\Sigma^* \vdash C$.
  We can show that $\Sigma^* \vdash B \vee C$ iff $\Sigma^* \vdash B$ or $\Sigma^* \vdash C$.

- Inductive case 4: $A = B \rightarrow C$.
  Induction hypotheses: $B^t = 1$ iff $\Sigma^* \vdash B$. $C^t = 1$ iff $\Sigma^* \vdash C$.
  We can show that $\Sigma^* \vdash (B \rightarrow C)$ iff $\Sigma^* \vdash B$ implies $\Sigma^* \vdash C$.

- Inductive case 5: $A = B \leftrightarrow C$.
  Induction hypotheses: $B^t = 1$ iff $\Sigma^* \vdash B$. $C^t = 1$ iff $\Sigma^* \vdash C$.
  We can show that $\Sigma^* \vdash (B \leftrightarrow C)$ iff ($\Sigma^* \vdash B$ iff $\Sigma^* \vdash C$).

Revisiting the Learning Goals

By the end of this lecture, you should be able to

- Define the completeness of formal deduction.
- Define consistency and satisfiability.
- Prove properties of consistent and satisfiable sets based on their definitions.
- Reproduce the key steps of the proof of the completeness theorem.