Propositional Logic: Soundness of Formal Deduction

Alice Gao

Lecture 9

Learning Goals

By the end of this lecture, you should be able to

- Define the soundness of formal deduction.
- Prove that a tautological consequence holds using formal deduction and the soundness of formal deduction.
- Show that no formal deduction proof exists using the contrapositive of the soundness of formal deduction.

Tautological Consequence

Let Σ be a set of propositional formulas. Let A be a propositional formula.

$$\Sigma \models A$$

- Σ semantically implies A.
- A is a tautological consequence of Σ.
- For any truth valuation t, if every formula in Σ is true under t ($\Sigma^t = 1$), then A is also true under t ($A^t = 1$).

Several ways of proving a tautological consequence:
truth table, direct proof, a proof by contradiction, etc.

Formal Deduction

Let Σ be a set of propositional formulas. Let A be a propositional formula.

$$\Sigma \vdash A$$

- Σ formally proves A.
- There exists a proof which syntactically transforms the premises in Σ to produce the conclusion A.
- A formal proof is a syntactic manipulation of symbols and it can be checked mechanically.

Tautological Consequence v.s. Formal Deduction

$\Sigma \models A$ and $\Sigma \vdash A$ appear to be similar.
Ideally, we would like them to be equivalent. This could mean two properties:

1. If $\Sigma \vdash A$, then $\Sigma \models A$. (Soundness of formal deduction)
   If there exists a formal proof from Σ to A, then Σ tautologically implies A.
   (Everything I can formally prove is a tautological consequence.)
2. If $\Sigma \models A$, then $\Sigma \vdash A$. (Completeness of formal deduction)
   If Σ tautologically implies A, there exists a formal proof from Σ to A.
   (I can formally prove every tautological consequence.)

Soundness and Completeness of Formal Deduction

Theorem: Formal Deduction is both sound and complete.

Soundness of Formal Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

If $\Sigma \vdash \alpha$*, then* $\Sigma \models \alpha$

Completeness of Formal Deduction means that all logical consequences in propositional logic are provable in Formal Deduction. That is,

If $\Sigma \models \alpha$*, then* $\Sigma \vdash \alpha$

Other proof systems

- resolution
- axiomatic systems
- semantic tableaux
- intuitionistic logic: sound but not complete. e.g. it cannot prove $p \vee (\neg p)$
- any system plus $p \wedge (\neg p)$ as an axiom: not sound but complete.
  not sound because we can prove $p \wedge (\neg p)$ which is false.
  complete because we can prove anything with $p \wedge (\neg p)$ as an axiom.

Proving the soundness of formal deduction

We will prove this by structural induction on the proof for $\Sigma \vdash A$.

A proof is a recursive structure.

A proof either

- derives the conclusion without using any inference rule, or (Base case)
- derives the conclusion by applying a rule of formal deduction on a proof. (Inductive case)

Proof of the soundness of formal deduction

Theorem: For a set of propositional formulas Σ and a propositional formula A, if $\Sigma \vdash A$, then $\Sigma \models A$.

Proof: We prove this by structural induction on the proof for $\Sigma \vdash A$.

Base case: Assume that there is a proof for $\Sigma \vdash A$ where $A \in \Sigma$. Consider a truth valuation such that $\Sigma^t = 1$. Since $A \in \Sigma$, then $A^t = 1$. Thus, $\Sigma \models A$.

(To be continued)

Proof of the soundness of formal deduction

Induction step: Consider several cases for the last rule applied in the proof of $\Sigma \vdash A$. (There is one case for every rule of formal deduction.)

- Assume that the proof of $\Sigma \vdash A$ applies the rule $\wedge+$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash C$ and reaches the conclusion $\Sigma \vdash B \wedge C$.

  Let me prove this case for you.

  Induction hypotheses: Assume that $\Sigma \vDash B$ and $\Sigma \vDash C$.
  We need to prove that $\Sigma \vDash B \wedge C$.

  Consider a truth valuation t such that $\Sigma^t = 1$.
  By the induction hypotheses, $B^t = 1$ and $C^t = 1$.
  By the truth table of $\wedge$, $(B \wedge C)^t = 1$.
  Therefore, $\Sigma \vDash (B \wedge C)$.

  (To be continued)

Proof of the soundness of formal deduction

Induction step (continued):

- Assume that the proof of $\Sigma \vdash A$ applies the rule $\rightarrow -$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash (B \rightarrow C)$ and reaches the conclusion $\Sigma \vdash C$.

  Try proving this case yourself.

Applications of soundness and completeness

1. The following inference rule is called Disjunctive syllogism.

   $$\text{if } \Sigma \vdash \neg A \text{ and } \Sigma \vdash A \vee B, \text{then } \Sigma \vdash B.$$

   where A and B are well-formed propositional formulas.

   Prove that this inference rule is sound.
   That is, prove that if $\Sigma \vDash \neg A$ and $\Sigma \vDash A \vee B$, then $\Sigma \vDash B$.
2. Show that there does not exist a formal deduction proof for $p \vee q \vdash p$, where p and q are propositional variables.
3. Prove that $(A \rightarrow B) \nvdash (B \rightarrow A)$ where A and B are propositional formulas.

Applications of soundness and completeness

The following inference rule is called Disjunctive syllogism.

$$\text{if } \Sigma \vdash \neg A \text{ and } \Sigma \vdash A \vee B, \text{then } \Sigma \vdash B.$$

where A and B are well-formed propositional formulas.

Prove that this inference rule is sound.
That is, prove that if $\Sigma \models \neg A$ and $\Sigma \models A \vee B$, then $\Sigma \models B$.

Proof:
Consider a truth valuation t under which $\Sigma^t = 1$. Since $\Sigma \models (\neg A)$ and $\Sigma \models A \vee B$, we have that $(\neg A)^t = 1$ and $(A \vee B)^t = 1$. We need to show that $B^t = 1$.
By the truth table of $\neg$, since $(\neg A)^t = 1$, $A^t = 0$.
By the truth table of $\vee$, since $(A \vee B)^t = 1$, at least one of A and B is true under t. Since $A^t = 0$, then $B^t = 1$.
Therefore, $\Sigma \models B$ holds. QED

Applications of soundness and completeness

Show that there does not exist a formal proof for $p \vee q \vdash p$, where p and q are propositional variables.

Proof:
By the contrapositive of the soundness of formal deduction, if $p \vee q \not\models p$, then $p \vee q \nvdash p$. Consider the truth valuation t where $p^t = 0$ and $q^t = 1$. By the truth table of $\vee$, $(p \vee q)^t = 1$. Thus, $p \vee q \not\models p$. Therefore, $p \vee q \nvdash p$.
QED

Applications of soundness and completeness

Prove that $(A \to B) \nvdash (B \to A)$
where A and B are propositional formulas.

Proof:

By the contrapositive of the soundness of formal deduction, if $(A \to B) \not\models (B \to A)$, then $(A \to B) \nvdash (B \to A)$. We need to give a counterexample to show that $(A \to B) \not\models (B \to A)$.

Let $A = p$ and $B = q$. Consider the truth valuation where $p^t = 0$ and $q^t = 1$. By the truth table of $\to$, $(p \to q)^t = 1$ and $(q \to p)^t = 0$. Therefore, $(A \to B) \not\models (B \to A)$ and $(A \to B) \nvdash (B \to A)$.

QED

Revisiting the Learning Goals

By the end of this lecture, you should be able to

- Define the soundness of formal deduction.
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