Propositional Logic
Introduction and Syntax

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Lecture 2

Outline

Learning goals

Propositions and Connectives

Propositional Language

Revisiting the learning goals

Learning goals

By the end of the lecture, you should be able to

- Determine whether an English sentence is a proposition.
- Determine whether an English sentence is a simple or compound proposition.
- Determine whether a propositional formula is atomic and/or well-formed.
- Draw the parse tree of a well-formed propositional formula.
- Given a propositional formula with no parentheses, make it a well-formed formula by adding parentheses according to the precedence rules.

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Propositions

A proposition is a statement that is either true or false.

Meaningless statements, commands, and questions are not propositions.

CQ on proposition

Examples of propositions

- The sum of 3 and 5 is 8.
- The sum of 3 and 5 is 35.
- Goldbach's conjecture: Every even number greater than 2 is the sum of two prime numbers.

Examples of non-propositions

- Question: Where shall we go to eat?
- Command: Please pass the salt.
- Sentence fragment: The dogs in the park
- Non-sensical: Green ideas sleep furiously.
- Paradox: This sentence is false.

Compound and simple propositions

- A compound proposition is formed by means of logical connectives.

  The commonly used logical connectives are “not”, “and”, “or”, “if, then”, and “iff”.

- A simple proposition is not compound and cannot be further divided.

Interpreting a compound proposition

To interpret a compound proposition, we need to understand the meanings of the connectives.

Let A and B be arbitrary propositions.

We will use 1 and 0 to denote true and false respectively.

Negation

"Not A" is true if and only if A is false.

A	not A
1	0
0	1

Conjunction

"A and B" is true if and only if both A and B are true.

A	B	A and B
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

A	B	A or B
1	1	1
1	0	1
0	1	1
0	0	0

"Or" may be interpreted in two ways

- The inclusive sense of "A or B or both"
- The exclusive sense of "A or B but not both"

In mathematics, the inclusive sense of "or" is commonly used.

Implication

A	B	if A then B
1	1	1
1	0	0
0	1	1
0	0	1

The only circumstance in which "if A then B" is false is when A is true and B is false.

Whenever A is false, "if A then B" is vacuously true.

Equivalence

"A iff B" is the same as "if A then B, and if B then A".

iff is pronounced as if and only if.

A	B	A iff B
1	1	1
1	0	0
0	1	0
0	0	1

CQ on compound or simple propositions

Remarks on connectives

The arity of a connective:

- The negation is a unary connective. It only applies to one proposition.
- All other connectives are binary connectives. They apply to two propositions.

Is a connective symmetric?

- And, Or, and Equivalence are symmetric. The order of the two propositions does not affect the truth value of the compound proposition.
- Implication is not symmetric. If A then B, and if B then A have different truth values.

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Propositional language L^p

The propositional language L^p consists of three classes of symbols:

- Proposition symbols: p, q, r, $\dots$.
- Connective symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

Oral reading of logical connectives		
$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	(inclusive) disjunction
$\rightarrow$	if, then (imply)	implication
$\leftrightarrow$	iff (equivalent to)	equivalence

- Punctuation symbols: (and).

Expressions of L^p

- expressions are finite strings of symbols. Examples: p, pq, (r), $p\wedge \rightarrow q$ and $\neg(p \wedge q)$.
- The length of an expression is the number of occurrences of symbols in it.
- empty expression: an expression of length 0, denoted by λ.
- two expressions u and v are equal if they are of the same length and have the same symbols in the same order.
- an expression is read from left to right.

Expression terminologies

- uv denotes the result of concatenating two expressions u, v in this order. Note that $\lambda u = u\lambda = u$.
- v is a segment of u if $u = w_1 v w_2$ where u, v, w_1, w_2 are expressions.

  v is a proper segment of u if v is non-empty and $v \neq u$.

  If $u = vw$, where u, v, w are expressions, then v is an initial segment (prefix) of u. Similarly, w is a terminal segment (suffix) of u.

Atomic formulas

Definition ($Atom(L^p)$)

$Atom(L^p)$ is the set of expressions of L^p consisting of a proposition symbol only.

Well-formed propositional formulas

Definition ($Form(L^p)$)

An expression of L^p is a member of $Form(L^p)$ if and only if its being so follows from (1) - (3):

1. $Atom(L^p) \subseteq Form(L^p)$.
2. If $A \in Form(L^p)$, then $(\neg A) \in Form(L^p)$.
3. If $A, B \in Form(L^p)$, then $(A * B) \in Form(L^p)$ where $*$ is one of the four binary connectives.

Note that $Form(L^p)$ is the minimum set that satisfies the three conditions above.

CQ on the first symbol in a well-formed formula

CQ on well-formed propositional formulas

Example: Generating Formulas

The following expression is a formula.

$$((p \vee q) \rightarrow ((\neg p) \leftrightarrow (q \wedge r)))$$

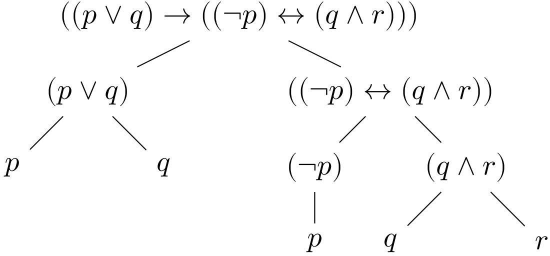
How is it generated using the definition of well-formed propositional formulas? One can use parse trees to analyze formulas.

Example: Parse Tree

Draw the parse tree for the following formula.

$$((p \vee q) \to ((\neg p) \leftrightarrow (q \wedge r)))$$

Parse tree:



Exercise: Parse Trees

Draw the parse tree for the following formula.

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

Precedence rules: for humans

Consider the following sequence of connectives:

$$\neg, \land, \lor, \to, \leftrightarrow$$

Each connective on the left has priority over those on the right.

Examples: Add back the brackets based on the precedence rules.

- $\neg p \lor q$
- $p \land q \lor r$
- $p \to q \leftrightarrow p$
- $\neg p \to p \land \neg q \lor r \leftrightarrow q$

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