CSC2411 - Linear Programming and Combinatorial Optimization*
Lecture 7: Ellipsoid Algorithm and Interior Point Methods

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Summary: This lecture provides an in-depth explanation of ellipsoid algorithm which was developed in 1979 by a Soviet mathematician L.G. Khachian. Ellipsoid algorithm is the first polynomial-time linear programming algorithm which also knows as Khachiyan's algorithm to acknowledge Khachiyan's discovery. This algorithm is fundamentally different from Simplex Algorithm in a sense that it does not exploit the combinatorial structure of linear programming and its based on binary search to determine whether a polyhedron is empty or not. We also present a high level description of interior point methods, which is the most commonly used algorithm for solving linear programming problems.

1 The Algorithm

The underlying idea of ellipsoid Algorithm is very simple and its based on an iterative procedure. Before, we provide a pseudocode of the algorithm we briefly describe its high level searching mechanism that used to solve the LP. At every iteration, we bound the solution of the problem within an ellipsoid and after every successive iteration we reduce the size of the ellipsoid. After enough iteration the algorithm either reports the solution or terminates due to that fact that ellipsoid has became too small to contain any solution. Below we present the ellipsoid algorithm and figure 1 illustrates the geometric interpretation of the algorithm at every iterations and figure 2 provides the intuition behind the ellipsoid transformation.

* Lecture Notes for a course given by Avner Magen, Dept. of Computer Sciecne, University of Toronto.

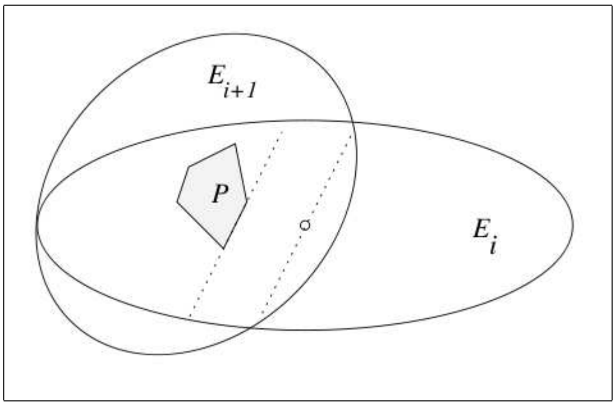

Figure 1: One iteration of the algorithm $E_i \to E_{i+1}$

Algorithm 1 Ellipsoid Algorithm

Input: An $m \times n$ system of linear strict inequalities $Ax < b$ of size L
Output: An n-vector x such that $Ax < b$, if exists; no otherwise (feasible/infeasible)
  Initialize:
  $VLB = 2^{-2nL-n^2}$ (Volume Lower Bound)
  $R = 2^{2L}$ (Radius of the initial ball)
  $E_0 = B(0, R)$ (Ball centered at 0 with Radius R)

  while Vol$(E_i) \geq VLB$ **do**
    if $y =$Center(E_i) is feasible by all of the m inequalities **then**
      report y; (declare feasible and stop)
    end if
    find a violation i such that $\langle a_i, y\rangle \geq b_i$ and Let $\frac{1}{2}E_i = E_i \cap \{x | \langle a_i, x\rangle \leq b_i\}$
      Construct ellipsoid E_{i+1} with the following properties:
        (i) $E_{i+1} \supset \frac{1}{2}E_i$
        (ii) Vol$(E_{i+1}) \leq e^{-\frac{1}{2n}} \times$ Vol(E_i)
  end while

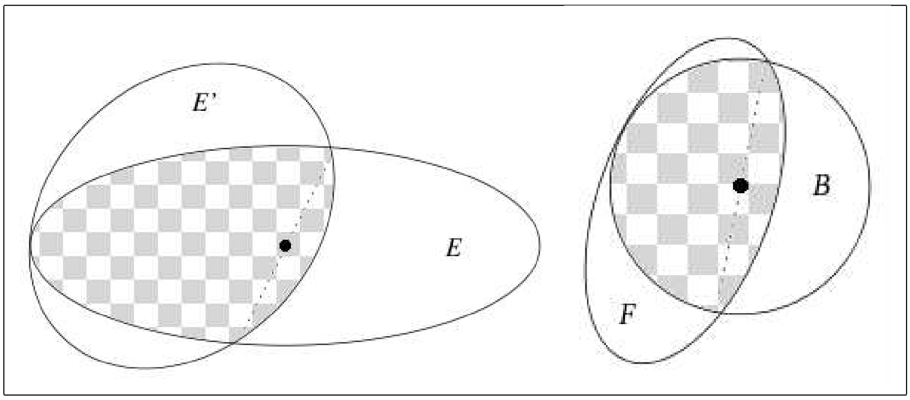

Figure 2: Original ellipsoid E and E' contains $\frac{1}{2}$ minimizing the volume (left hand side) $\Rightarrow$ Ellipsoid E transformed to unit ball, B, and the new E' contains northern half-ball of F (right hand side)

2 Constructing Ellipsoid at Every Iteration

Definition 2.1. Ellipsoid can be considered as an affine map of a unit ball $B(0,1)$ in $\mathbb{R}^n$.

$$\begin{aligned} E &= T(B) \\ &= \{T(x) |\ x \in B\} \\ &= \{Ax + c |\ \|x\| \leq 1\} \\ &= \{y |\ \|A^{-1}(y-c)\| \leq 1\} \\ &= \{y | (y-c)^t (A^{-1})^t A^{-1} (y-c) \leq 1\} \\ &= \{y | (y-c)^t Q^{-1} (y-c) \leq 1\}\ where\ Q = AA^t \end{aligned} \tag{1}$$

Q is $n \times n$ symmetric matrix which is positive definite, that is $\forall x \in \mathbb{R}^n$ and $x \neq 0, x^t Q x > 0$

Theorem 2.2. *For an ellipsoid B and E where $E = T(B)$, T is an affine transformation such that $T(x) = Ax + c,\ x \in B$, then* $\mathrm{Vol}(E) = \mathrm{Vol}(T(B)) = \sqrt{\det(Q)} \cdot \mathrm{Vol}(B)$.

Proof First we are you going to simplify $\mathrm{Vol}(E)$ expression:

$$\begin{aligned} \mathrm{Vol}(E) &= \mathrm{Vol}(T(B)) \\ &= \sqrt{\det(Q)} \cdot \mathrm{Vol}(B) \\ &= \sqrt{\det(AA^t)} \cdot \mathrm{Vol}(B) \\ &= |\det(A)| \cdot \mathrm{Vol}(B) \end{aligned} \tag{2}$$

The general ellipsoid can be expressed as follows:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = b$$

and the volume is calculated as follows:

$$\frac{4}{3}\pi abc$$

The parametric equation of ellipsoid can be written as follows (for sphere we have $a = b = c$):

$$\begin{aligned} a &= \tfrac{x_1}{\cos\theta\sin\phi} \\ b &= \tfrac{x_2}{\sin\theta\sin\phi} \\ c &= \tfrac{x_3}{\cos\phi} \end{aligned}$$

Let λ be the Eigenvalues of matrix A. Then we have that $Ax = \lambda x$ and $\det(\lambda I) = \det(A)$. After transforming B using $T(B) = Ax + c$, then we can re-write the parametric equation as follows. Also, without the loss of generality we can assume $c = 0$.

$$\begin{aligned} a' &= \tfrac{\lambda_1 x_1}{\cos\theta\sin\phi} \\ b' &= \tfrac{\lambda_2 x_2}{\sin\theta\sin\phi} \\ c' &= \tfrac{\lambda_3 x_3}{\cos\phi} \end{aligned}$$

To complete the proof we show how to calculate the new volume of B.

$$\begin{aligned} \mathrm{Vol}(E) &= \frac{4}{3}\pi a'b'c' \\ &= \frac{4}{3}\pi\lambda_1\lambda_2\lambda_3 abc \\ &= |\det(A)| \cdot \mathrm{Vol}(B) \end{aligned} \tag{3}$$

Theorem 2.3. *(Löwner John) If $K \subset \mathbb{R}^n$ is a convex body, then there is a unique ellipsoid E of minimum volume with the following property $E \supset K$ and $\frac{1}{n}E \subset K$.*

Remark: If K is symmetric around the origin then the same holds with $\frac{1}{\sqrt{n}}$ replaced by $\frac{1}{n}$

Claim we may assume the following:

- $E = B(0, 1)$
- $N = \frac{1}{2}E = \{x| \, \|x\| \leq 1; x_n \geq 0\}$

Proof apply an affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and $T(x) = Ax + c$, so that we have:

- Given an ellipsoid its affine transformation to sphere is $T^{-1}(E) = B(0, 1)$
- $T(\frac{1}{2}E) = N$
- Vol($T(K)$) =Vol(K) $\times \, |\det(T)|, \; K \in \mathbb{R}^n$ by Theorem 2.2

In figure: 2 we transform the original ellipsoid E to the unit ball B and the minimal volume of E' is transformed to F. Its important that ratio between the original ellipsoid, E and E' is equal to the ratio between the newly transformed unit ball B and the ellipsoid F, so we maintain the ratio $\frac{\mathrm{Vol}(E)}{\mathrm{Vol}(E')} = \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(F)}$.

Proof:

- $T^{-1}(E) = B \Rightarrow \mathrm{Vol}(B) = |\det T^{-1}|\mathrm{Vol}(E)$ by Theorem 2.2
- $T^{-1}(E') = F \Rightarrow \mathrm{Vol}(F) = |\det T^{-1}|\mathrm{Vol}(E')$ by Theorem 2.2
- $\frac{\mathrm{Vol}(B)}{\mathrm{Vol}(F)} = \frac{|\det T^{-1}|\mathrm{Vol}(E)}{|\det T^{-1}|\mathrm{Vol}(E')} = \frac{\mathrm{Vol}(E)}{\mathrm{Vol}(E')}$

Let F be the new ellipsoid after every iteration now we provide the details of how to find such an ellipsoid.

- $c = (0, 0, ..., \gamma_{>0})$, where c is the center of the ellipsoid
- the ellipsoid will be axis aligned, Q is diagonal such that
  - $F = \{x|(x-c)^t Q^{-1}(x-c) \leq 1\}$
  - Vol$(F) = \det Q$
  - $$Q = \begin{pmatrix} \alpha & & & & \\ & \alpha & & & \\ & & \alpha & & \\ & & & \cdots & \\ & & & & \beta \end{pmatrix} \alpha, \beta > 0$$
  - $((0, 0, \cdots, 1) - c)^t Q{-}1((0, 0, \cdots, 1) - c) = 1$
    - $(x-c)^t Q^{-1}(x-c) = 1$
    - $(1-\alpha)^2\beta^{-1} = 1$, $\alpha^{-1} + \beta^{-1}\gamma^2 = 1$

Therefore, we get the following equalities:

$$\min\ Vol(F) = \min\ \det(Q) = \min\ \alpha^{n-1}\beta$$

base on the below parameters:

- $\alpha = \frac{n^2}{n-1}$
- $\beta = \frac{n^2}{(n+1)^2}$
- $\gamma = \frac{1}{n+1}$

Thus, we can construct an upper bound on the of Vol(F) by optimizing $\alpha^{n-1}\beta$:

$$\begin{aligned} Vol(F) &= \alpha^{n-1}\beta \\ &= (1 + \frac{1}{n^2-1})^{n-1}(1 - \frac{2n+1}{(n+1)^2}) \\ &\leq e^{\frac{1}{n-1}} \times e^{-\frac{2}{n+1}} \\ &\leq e^{-\frac{1}{n-1}} \end{aligned} \tag{4}$$

In addition, the factor shrinkage would be $\sqrt{\det(Q)} = e^{-\frac{1}{2(n+1)}}$. Now, in order to remove restriction on B as a ball, we can generalize it to any ellipsoid such that $E_i = E(Q_i, c_i) \rightarrow E_{i+1} = E(Qi + 1, ci + 1)$, where

$$Q_{i+1} = \tfrac{n^2}{n^2}(Q - \tfrac{2}{n+1}\upsilon\upsilon^t), \text{ where } \upsilon\upsilon^t \text{ is the matrix } a_{ij} = \upsilon \cdot \upsilon$$
$$ci + 1 = c_i - \tfrac{1}{n-1}\upsilon,$$
$$\upsilon = \tfrac{Q_i a}{\sqrt{a^t Q_i a}}$$

3 Algorithm Oracles

The goal is to solve the optimization problem such as ellipsoid algorithm through a polynomial oracle. Ellipsoid algorithm is based on the *Membership Oracle* and *Separation Oracle*.

Definition 3.1. Membership Oracle given x, efficiently determine if x is feasible or not.

Definition 3.2. Separation Oracle: Given y that is not feasible, supply efficiently a vector $a \neq 0$ and b so that:

1. $(a, y) = b$
2. $\forall$ feasible x, $(a, x) \leq b$

Observation: Can LP with n variables and m constraints be solved using the above oracles as efficient as function of m and n. Not only ellipsoid can solve LP in polynomial time but it can also maintain its efficiency in the presence of a large number of constraints.

4 LP Reduction

Consider Multicut problem that can be reduced to an LP with exponential number of constraint and we can solve it by ellipsoid algorithm.

Input: Graph(V,E) costs $c_e \geq 0, \forall e \in E$
k terminal pairs (s_i, t_i)

Output: a min-cost multicut, i.e. a set of edges of min total weight, the removal of which disconnect all the pairs

IP Formulation:

$$\min \sum c_e x_e$$
$$x_e \in \{0, 1\}$$
$$\sum_{e \in \pi} x_e \geq 1$$

Intuition: $\forall i, \forall \pi$ in the path between s_i and t_i we pick at least on edge.

Claim: we do not know how to deal with the constraint $x_e \in \{0, 1\}$ and we relax it by using $x_e \in [0, 1]$ instead. However, when we perform the relaxation its not longer obvious that LP can be solved because we have exponentially many constraints. Now, we show that the relaxation of the integer program to linear programming has a membership and a separation oracle.

Proof: given x,
if $\exists e\ x_e \notin [0, 1]$ clearly we obtain the oracles.
Now, assume $x \in [0, 1]^E$

$\forall i$ check if the $\mathrm{dist}(s_i, t_i)$ as defined by x_e is smaller than 1.
If this condition does not hold $\forall i$, then return 'YES'.

otherwise $\exists i$ so that $\mathrm{dist}(s_i, t_i) < 1 \Rightarrow \exists \pi$ between s_i, t_i and $\sum_{e \in \pi} x_e < 1$ that is the separation oracle.

5 Performance Analysis

Despite the fact that ellipsoid is a polynomial algorithm its not commonly used to practically solve LP problems, mainly because it has to work with irrational numbers such as $\sqrt{a_i^t Q_i a_i}$ and to deal with their rounding related issues. At every iteration of ellipsoid algorithm we must maintain the Q to be positive definite and we have to rounded the ellipsoid such that it contains $\frac{1}{2}E_i$. Also, we must be very careful when we are rounding the ellipsoid because the new volume should not be significantly different from the original volume of the ellipsoid. In addition, we accumulate the rounding error at every iteration which could exponentially increases the error.

In the table below we summarize the characteristics of each algorithm. Its important to note that, despite the fact the ellipsoid algorithm theoretically has a polynomial time complexity but in the practice the simplex method outperforms it. However, as we saw in the previous example the ellipsoid is very extensible and can be used to solve many different types of problems. Clearly, the interior point method is superior in every aspects, both theoretically and practically.

	Theoretically	**Practically**	**Extensibility**
Simplex	$\times$	$\surd$	$\times$
Ellipsoid	$\surd$	$\times$	$\surd$
Interior Point Methods	$\surd$	$\surd$	$\surd$

6 Interior Point Methods

Idea: Interior points is an iterative method similar to simplex algorithm, where it always holds a feasible solution and it attempts to improve in two different phases. But, unlike the simplex it moves in the interior of the feasible set.

Definition: P is a feasible region,
x is 'almost optimal' if $\langle x, c \rangle \leq OPT(P) - 2^{-2L}$

Lemma: If x is almost optimal and if y is a BFS so that $\langle y, c \rangle \leq \langle x, c \rangle$ then y is optimal.

Proof: If y' is a BFS $\langle y', c \rangle < \langle y, c \rangle$ then $\langle y, c \rangle - \langle y', c \rangle > 2^{-2L}$

References