CSC2411 - Linear Programming and Combinatorial Optimization*
Lecture 11: Primal-Dual Schema and Algorithms

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Summary: This lecture starts with an LP relaxation of the max sat problem and a proof showing how close the relaxed optimum is to the integral optimum. Then we move on to a definition and proof complementary slackness and use it to establish primal-dual schema and algorithms. We then show an example of using such an algorithm on the set cover problem.

1 Max Sat

The maximum satisfiability problem asks for the maximum number of conjunctive normal form (CNF) clauses that can be satisfied by any assignment of the variables involved:

Input: CNF formula (e.g. $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_1) \wedge (\neg x_3 \vee \neg x_1)$)

Output: Truth assignments satisfying maximum number of clauses(e.g. from above: $x_1 = 1; x_2 = 1; x_3 = 0$)

1.1 Integer Program

$$\max \sum_{C_j} Z_{C_j}$$

$$s.t \sum_{i \in C^+} x_i + \sum_{i \in C^-} (1 - x_i) \geq Z_{C_j}$$

$$x_j \in \{0, 1\}$$

$$Z_{C_j} \in \{0, 1\}$$

* Lecture Notes for a course given by Avner Magen, Dept. of Computer Sciecne, University of Toronto.

1.2 LP Relaxation

$\max \sum_{C} Z_C$

$s.t \sum_{i \in C^+} x_i + \sum_{i \in C^-} (1 - x_i) \geq Z_{C_j}$

$0 \leq x_i \leq 1$

$Z_{C_j} \leq 1$

Rounding procedure: Assign each variable to be true with probability equal to the corresponding value, x_i, picked by the above LP.

We now show that the above optimum is not much worse than the integral optimum. To do so, we ask the following question:

What is the probability that a certain clause, C, is not satisfied?

Let (x,z) be a solution to the LP and B_{C_j} be the probability that clause C_j with K variables is not satisfied. Then

$$Pr[B_{C_j}] = \prod_i (1 - x_i^*), \text{ where } x_i^* = \begin{cases} x_i & \text{if } i \in C^+ \\ 1 - x_i & \text{if } i \in C^-. \end{cases}$$

by the geometric means inequality, $Pr[B_{C_j}] \leq \left(\frac{\sum (1 - x_i^*)}{K} \right)^K$

$= \left(1 - \frac{\sum x_i^*)}{K}\right)^K \leq \left(1 - \frac{Z_{C_j}}{K}\right)^K$

$Pr[\text{ satisfying } C_j] = 1 - Pr[B_{C_j}]$

$\geq 1 = \left(1 - \frac{Z_{C_j}}{K}\right)^K \geq Z_C \left(1 - \left(1 - \frac{1}{K}\right)^K\right)$

$\geq Z_C \left(1 - e^{-1}\right)$

$\Rightarrow E[\text{ \# of satisfied clauses }] = \sum_j Pr[\text{ satisfying } C_j]$

$\geq \sum Z_{C_j} \left(1 - \frac{1}{e}\right) = \text{OPT}^* \left(1 - \frac{1}{e}\right)$

$\geq \text{OPT} \left(1 - \frac{1}{e}\right)$

2 Primal-Dual Schema

Before describing the algorithm in detail, we must establish an important result:

2.1 Complementary Slackness

Consider the following primal and dual pair:

Primal:

$$\begin{gathered} \min < c, x > \\ \text{s.t Ax} \geq \text{b} \\ \text{x} \geq 0 \end{gathered}$$

Dual:

$$\max < b, y >$$
$$\text{s.t } y \geq 0$$
$$yA \leq c$$

At an optimum, each variable in the above inequalities is either zero or the corresponding inequality is satisfied as an equality:

Definition 2.1. Complementary Slackness (CS) states that x,y is a Primal-Dual optimum iff

$\forall j$, $x_j = 0$ or $(yA)_j = c_j$ (Primal Complementary Slackness)
$\forall i$, $y_i = 0$ or $(Ax)_i = b_i$ (Dual Complementary Slackness)

Proof: By strong duality, at an optimum, $< c, x > = < yA, x >$, which implies that $\forall x_i \neq 0, c_i = (yA)_i$ (Primal CS). Also by strong duality, $< y, Ax > = < b, y >$, which implies that $\forall y_j \neq 0, b_j = (Ax)_j$ (Dual CS).

Our algorithm will try to narrow the gap between the value of the objective functions of the primal and dual. It will correct values of x and y locally based on complementary slackness and stops when an integrally feasible solution with a small primal-dual gap is reached.

2.2 Relaxed Complementary Slackness

We can make CS more useful to our algorithm by relaxing some of the equalities:

Primal relaxed CS: $\forall j$, $x_j = 0$ or $\frac{C_j}{\alpha} \leq (yA)_j \leq C_j$
Dual relaxed CS: $\forall i$, $y_i = 0$ or $b_i \leq (Ax)_i \leq \beta b_i$

Where $\alpha \geq 1$ and $\beta \geq 1$.

Claim 2.2. *If x,y satisfies the CS conditions, then* $< x, c > \leq \alpha\beta < y, b >$.

Proof:
$$< x, c > = \sum_j (yA)_j x_j$$
$$= \sum_{j, x_j > 0} (yA)_j x_j$$
$$\geq \sum_{j, x_j > 0} \frac{C_j}{\alpha} x_j$$
$$= \frac{1}{\alpha} < c, x >$$
$$\Rightarrow < c, x > \leq \alpha < y, Ax >$$

Similarly, it can be shown that $< y, b > \geq \frac{1}{\beta} < yA, x >$, completing the proof.

We now have all the tools needed for our algorithm, which follows.

2.3 Algorithm Outline

We start with the primal and dual versions of our problem:

Primal:

$$\text{min} < c, x >$$
$$\text{s.t Ax} \geq \text{b}$$
$$\text{x} \geq 0$$

Dual:

$$\text{max} < y, b >$$
$$\text{s.t yA} \leq \text{c}$$
$$\text{y} \geq 0$$

where we assume that $c \geq 0$ and $A \geq 0$.

Algorithm:

Initialize with $x = 0, y = 0$
While x not feasible {
Raise y's that are not involved in tight inequalities ***(here, tight means that the inequality is satisfied as a strict equality)*** $yA \leq c$ until one or more inequalities become tight }

2.4 An Example: Set Cover

The LP relaxation of the set cover problem is the following:

Primal:

$$\text{min} \sum C_s X_s$$
$$Ax \geq 1$$

Dual:

$$\text{max} \sum y_s X_s$$
$$y_e \geq 0$$
$$yA \leq C$$

Lets assume that every element belongs to at most f sets.

The primal-dual algorithm that we now construct will get to an integral primal-dual pair that will satisfy the relaxed CS conditions with $\alpha = 1, \beta = f$.

Relaxed CS states that

$\forall s, x_s \neq 0 \Rightarrow \sum_{e \in S} y_e = (yA)_s = C_s$
and also
$\forall e, y_s \neq 0 \Rightarrow \sum_{S \ni e} x_s \leq f$

Algorithm: Approximation to set cover with factor f

Initialize with $x = 0, y = 0$
do until all elements are covered $\{$
pick an uncovered element, e, and raise y_e until the constraint on one or more sets become tight
pick all tight sets and add to cover declare elements in those sets as covered $\}$

The above algorithm works since if e is not covered, containing sets are not picked. Therefore, we know that if a set is not covered, the corresponding y must have some slack, which implies that y_e can be increased without violating constraints.