1. Let \((X, d)\) be a metric space with \(d(x, y) \in \{1, 2\}\), for every two distinct \(x, y \in X\). Moreover for every \(x \in X\) we have \(|\{z : d(x, z) = 1\}| < B\) for some number \(B\). Show that there is an isometric embedding of \(X\) into \(\ell_\infty^{O(B \log n)}\). (Hint: Look for a random Frechet embedding.)

2. Let \(P_n\) be a path of length \(n\) (so the distance between \(i\) and \(j\) is \(|i - j|\)). Prove that for every \(D > 1\) and \(\epsilon > 0\) there exists an \(n = n(D, \epsilon)\) such that whenever \(f\) embeds \(P_n\) into a metric \(d\) with distortion at most \(D\), there are \(a < b < c\) with \(b = \frac{a + c}{2}\) such that \(f\) restricted to the subspace \(\{a, b, c\}\) of \(P_n\) is an embedding with distortion at most \(1 + \epsilon\).

3. (a) Let \(Q_m\) be the hypercube \(\{0, 1\}^m\) equipped with the Hamming distance. Assume \(f : Q_m \to \ell_2\) is a nonexpanding embedding that satisfies the following average condition

\[
\sum_{a, b \in Q_m} \|f(a) - f(b)\|_2^2 \geq \frac{1}{D} \cdot \sum_{a, b \in Q_m} d_{Q_m}^2(a, b)
\]

Prove that \(D = \Omega(m)\).

(b) Use part (a) to prove that the main structure theorem of ARV is tight (i.e., separation of \(1/o(\sqrt{\log n})\) is not possible in some cases).
4. Recall that in non-uniform sparsest cut, we have to find a set $S \subset \{1, \ldots, n\}$ so as to minimize
\[
\frac{\sum_{i,j} \gamma_{ij} \delta_S(i,j)}{\sum_{i,j} \eta_{ij} \delta_S(i,j)}
\]
where $\gamma_{ij}$ and $\eta_{ij}$ are nonnegative numbers. (For the problem of uniform sparsest cut, $\gamma_{ij} = 1$ if $ij \in E$, and 0 otherwise, whereas $\eta_{ij} = 1$ always.)

The following is an SDP relaxation to the problem (verify for yourself).

\[
\text{minimize} \quad \sum \gamma_{ij} \|v_i - v_j\|_2^2 \\
\text{subject to} \quad \sum \eta_{ij} \|v_i - v_j\|_2^2 = 1 \\
\|v_i - v_j\|_2^2 \leq \|v_i - v_k\|_2^2 + \|v_k - v_j\|_2^2 \quad \forall i, j, k.
\]

Suppose that the edit distance metric on $\{0,1\}^m$ is embeddable with distortion $\alpha_m$ into an $\ell_2$ metric. Prove that under this assumption there is an instance of non-uniform sparsest cut so that the above SDP has integrality gap (namely the ratio of the solution to the original problem and the solution to the SDP) at least $\Omega(\frac{\log \log n}{\alpha_m \log n})$.

5. Consider a $\beta$-Lipschitz\(^1\) function $f : S^{n-1} \to \mathbb{R}^+$. Use Levy’s lemma to prove that for $m < \min\{n, \frac{2^n}{(2m)^{1/2}} - 2\}$, there exists an isometric embedding $g : E_m \to S^{n-1}$ such that $|f(g(x)) - M_f| \leq \epsilon$ for every $x \in E_m$. Here, $E_m$ is $\{-1/\sqrt{m}, 1/\sqrt{m}\}^m$ equipped with the $\ell_2$ norm, and $M_f$ is the median of $f$.

\(^1\)i.e. $\forall x, y, |f(x) - f(y)| \leq \beta\|x - y\|_2$. 

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