CSC2414 - Metric Embeddings[*]
Lecture 6: Reductions that preserve volumes and distance to affine spaces $\&$ Lower bound techniques for distortion when embedding into ℓ_2

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Summary: According to Johnson-Lindenstrauss Lemma there is a projection from a Euclidian space to a subspace of dimension $O(\frac{\log n}{\epsilon^2})$, that scales distances within a factor of $1+\epsilon$. A natural extension of this result suggests the preservation of other geometric characteristics like angles, areas and volumes of simplexes spanned by many vectors. In this direction we see how to obtain similar results when our concern is the preservation of general geometric characteristics.

On the other hand we have stated that certain metrics (such as C_4) cannot be isometrically embedded into ℓ_2. We make this fact more concrete by introducing a class of inequalities (Poincaré inequalities) that provide a technique for proving lower bounds for the required distortions. We also apply this result to the hypercube Q_n and obtain a $\sqrt{n}$ lower bound.

1 Reductions that preserve angles and volumes

In last lecture we saw a lemma by Johnson and Lindenstrauss that allows us to embed an n-point metric $d \in \ell_2$ to a $O(\frac{\log n}{\epsilon^2})$ dimensional Euclidean space with distortion $1+\epsilon$. The idea was to project the original space onto a random $\frac{\log n}{\epsilon^2}$-dimensional subspace of the original space, suggesting a probabilistic algorithm for producing the low-distortion embedding.

Theorem 1.1. *(Johnson-Lindenstrauss) For any $\epsilon > 0$, any n-point ℓ_2 metric can be $(1+\epsilon)$-embedded into $\ell_2^{O\left(\frac{\log n}{\epsilon^2}\right)}$.*

A natural question to ask is whether we can do any better with the dimension. A negative result by Alon (unpublished manuscript) shows that if the $n+1$ points

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$0, e_1, \ldots, e_n \in \mathbb{R}^n$ (where the e_i's are the standard orthonormal basis) are $(1+\epsilon)$-embedded into ℓ_2^k, where $100n^{-1/2} \leq \epsilon \leq \frac{1}{2}$ then $k = \Omega\left(\frac{1}{\epsilon^2 \log \frac{1}{\epsilon}} \log n\right)$. In other words the bound of Theorem 1.1 is almost tight.

The previous bounds concern embeddings that preserve the pairwise distances which is one of many characteristics of a subset in the Euclidean space. Some other characteristics include center of gravity, angles defined by triplets, volumes of sets, distances between points to lines or even higher dimensional affine spaces [Mag02]. It is easy to see that these characteristics are independent. For example, a function that "almost" preserves the distance between any two points can affect dramatically the angles defined by triplets as shown in Figure 1. Additionally, we can note that since the three points that define a right angle are mapped to three "almost" collinear points, even the areas of triangles cannot be preserved.

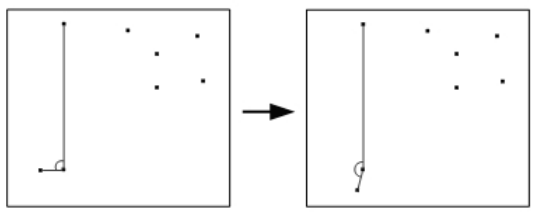

Figure 1: A low distortion embedding that does not preserve angles.

The new challenge is to generalize the notion of a good embedding that preserves pairwise distances, to a dimension reduction that preserves distances, angles and volumes of simplexes spanned by k points (for more details we refer to [Mag02]). Of course such an embedding might be more demanding regarding the lower bound of the dimension that it needs to guarantee good distortion.

First, one has to consider good dimension reductions with respect to area preservation. Keeping in mind that this can be generalized to a dimension reduction that preserves the volume of k-dimensional simplexes we can expect that k needs to appear in the lower bound of the dimension.

Now consider the problem of determining a low-dimensional embedding that preserves pairwise distances, areas of triangles and distance of points from lines. Note here that the preservation of these characteristics is strongly related to the low distortion on heights of triangles. So, in order to extract the properties of a good dimensional reduction for these characteristics, it is useful to look for specific instances where the distance-preservation implies low distortion on heights. In the next lemma, our restriction to 2 dimension hardly affects its validity for higher dimensions.

Lemma 1.2. *Let $A, B, C \in \mathbb{R}^n$ be the vertices of a right angle isosceles, where the right angle is at A, and let f be a contracting embedding of its vertices into a Euclidean space with distortion $1+\epsilon, \epsilon < \frac{1}{6}$. Let also $h = \|A - B\|, b = C - A, c = B - A$ and let h' be the height that corresponds to $f(B)$. Then*

$$|\langle b,c\rangle| \le 2\epsilon h^2$$
$$\frac{h}{1+2\epsilon} \le h' \le h$$

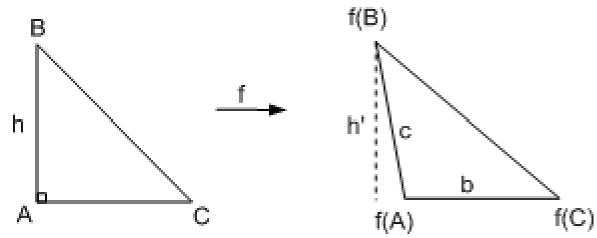

Figure 2: The triangle that preserves heights

Proof. Recall that f is non expanding with contraction at most $1+\epsilon$. It is easy to see that $|\langle b,c\rangle| = \left|\|b\|^2 + \|c\|^2 - \|b-c\|^2\right|/2$, is maximized when $\|b\|^2 = \|c\|^2 = \frac{h}{1+\epsilon}$. For these values $\|b-c\|^2 = \sqrt{2}h$. Combining this with the previous observation we obtain $|\langle b,c\rangle| \le 2\epsilon h^2$.

On the other hand the same values of $\|b\|, \|c\|$ maximize $|\cos(\theta)|$, where $\theta = \angle(b,c)$. More specifically $|\cos(\theta)| \le 2\epsilon + \epsilon^2$ and since $\epsilon \le \frac{1}{6}$, we get $|\sin(\theta)| > \frac{1+\epsilon}{1+2\epsilon}$. Hence

$$\frac{h}{1+2\epsilon} = \frac{h}{1+\epsilon}\frac{1+\epsilon}{1+2\epsilon} \le h' \le c\sin\theta \le h$$

□

The idea is now to enrich the stability of any triangle by considering a right isosceles for any of the edges of the triangle. For example, consider a triangle that is formed by the vectors u, v, z. For the edge uv, and for any other edge respectively, we will introduce a right isosceles with edge defined by the height corresponding to uv. So let u' be the projection of z to the affine hull of $\{u, v\}$ (i.e. the line u, v).Let also v' be a vector such that u, v', v are collinear and $||z - v'|| = ||v' - u'||$ (Figure 3). We will refer to this as the stabilization of a space. Clearly then, if we are able to preserve the pairwise distances within the pairs of all vectors, then the distortion on the angles and the areas will be low.

If the original space has n points, we can stabilize it with a total of $6\binom{n}{3}$ points. What we have in mind is to apply Theorem 1.1 to the stabilized space. Clearly, the new bound for the dimension is at most $\epsilon^{-2}\log(n + 6\binom{n}{3}) \approx 3\epsilon^{-2}\log n$.

Till now, our concern was to preserve areas or in other words the volume of a simplex spanned by three vectors (note here that the choice of $\binom{n}{3}$ vectors might not

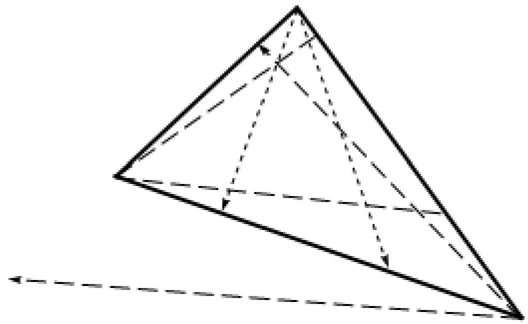

Figure 3: The stabilization of a triangle

be just a coincidence). Clearly the number of additional points that we need in order to preserve higher dimensional volumes of simplexes defined by k points, should depend on k. Just to get a flavor, and for simplexes defined by 4 points, we can define an analog of a right isosceles triangle (see Figure 4) that has good behavior when embedded with low distortion.

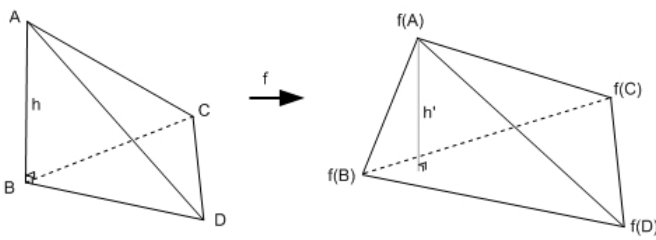

Figure 4: Simplexes defined by 4 points that preserve their volume

Note here that the initial space could live in $\mathbb{R}^m$. As far as we are interested in preserving the volume of simplexes defined by k vectors, we can treat them as if they lived in $\mathbb{R}^{k-1}$. We can introduce an appropriate notation that will be useful for volumes of simplexes of higher dimensions. Having in mind the analogies (see Figure 4), and now for a simplex that is defined by 4 vectors A, B, C, D, we introduce a new simplex with the same height h and with edges $||K - M|| = ||K - L|| = h$, while the vectors $K - L, K - M$ define a right angle (see Figure 5). We can think of h as the norm of the vector $A - P(A, \{B, C, D\})$, where $P(A, \{B, C, D\})$ is the projection of A to the affine hull $\mathcal{L}(S) = \{\sum_{i=1}^{|S|} \lambda_i \alpha_i : a_i \in S, \lambda_i \geq 0\}$ of $S = \{B, C, D\}$. Then if we denote by $ad(x, S)$ the distance of x to $\mathcal{L}(S)$, and by $r_1, \ldots, r_{|S|-1}$ a collection of orthonormal vectors of $\mathcal{L}(S)$, then $L = P(A, \{B, C, D\}) + ad(A, \{B, C, D\}) \cdot r_1$ and $K = P(A, \{B, C, D\}) + ad(A, \{B, C, D\}) \cdot r_2$.

For the simplex defined by 4 points above we introduce $\Theta(\binom{n}{4})$ vectors to preserve its volume. For extending the previous construction to simplexes defined by k vectors

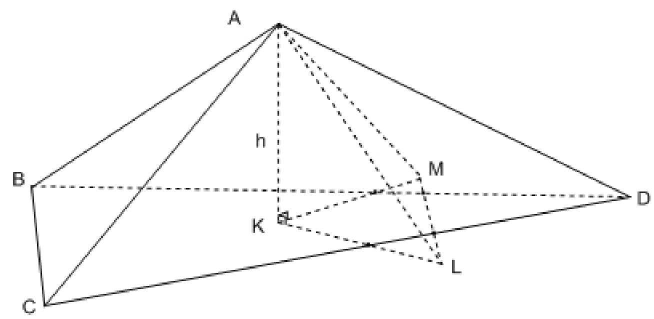

Figure 5: The stabilization of the tetrahedron $ABCD$

one has to be careful with the relation between the number of points of the initial space and the value of the distortion. However, in the general case it suffices to introduce $\Theta(\binom{n}{k})$ vectors in order to preserve the volume of k-dimensional simplexes giving rise to the following corollary of Theorem 1.1.

Corollary 1.3. *For any $\epsilon > 0$, any n-point ℓ_2 metric can be $(1+\epsilon)$-embedded in $\ell_2^{O\left(\frac{k \log n}{\epsilon^2}\right)}$ so that the embedding also preserves the volume of simplexes defined by k points.*

2 Lower bounds for the distortion when embedding into ℓ_2

We are interested in providing lower bounds for the required distortion for embeddings into ℓ_2. So far we have seen that n-points metric spaces can be embedded into ℓ_2 with distortion $O(\log n)$ (Bourgain's Theorem). Here we provide a weaker lower bound for the cube $\{0,1\}^m$. More specifically we prove that the cube $\{0,1\}^m$ (with $n = 2^m$), equipped with the Hamming distance (or ℓ_1), suffers distortion $\Omega(\sqrt{\log n})$ to be embedded into ℓ_2.

As a motivating example we can refer to the 2-dimensional cube Q_2 (4-cycle C_4) which cannot be isometrically ebmedded into ℓ_2. The reason is that a mapping that preserves the distances for the edges has to expand the diagonals. Actually, the distortion we have to suffer is $\sqrt{2}$, something that as we will see is related to the fact that the cube is 2-dimensional. The main idea for this is captured by the following lemma [Mat02]. The lemma says that the sum of the squares of the diagonals of a quadrilateral is at most the sum of the squares of its edges. Note here that for our convenience we will work with $||\cdot||^2$ instead of $||\cdot||$.

Lemma 2.1. *(Short diagonal inequality) Let x_1, x_2, x_3, x_4 be arbitrary points in a Euclidean space. Then*

$$||x_1 - x_3||^2 + ||x_2 - x_4||^2 \leq ||x_1 - x_2||^2 + ||x_2 - x_3||^2 + ||x_3 - x_4||^2 + ||x_4 - x_1||^2 \tag{1}$$

Proof. If $x_1, x_2, x_3, x_4 \in \mathbb{R}^m$ then $||x_i||^2 = \sum_{j=1}^m |x_i(j)|^2$ and therefore in order to prove (1), it suffices to show that this is valid for each of the coordinates i.e. work in $\mathbb{R}$. Trivially then we just observe that for any reals a, b, c, d the following is true

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 - (a-c)^2 - (b-d)^2 = (a-b+c-d)^2 \geq 0.$$

$\square$

The previous lemma can be directly applied to the Hamming cube $Q_2 = \{0,1\}^2$.

Corollary 2.2. *The cube Q_2 cannot be embedded into ℓ_2 with distortion less than $\sqrt{2}$.*

Proof. We can assume w.l.o.g. that if $Q_2 \overset{D}{\hookrightarrow} \ell_2$ then the embedding is contracting only; that is, for any $i \in Q_2$, if x_i is its image in ℓ_2 then

$$\|x_i - x_j\|_2 \leq d_H(i,j) \leq D\|x_i - x_j\|_2 \qquad \forall i,j \in Q_2. \tag{2}$$

Now note that $d_H^2(1,2) + d_H^2(2,3) + d_H^2(3,4) + d_H^2(4,1) = 4$ and $d_H^2(1,3) + d_H^2(2,4) = 8$, or in other words

$$d_H^2(1,2) + d_H^2(2,3) + d_H^2(3,4) + d_H^2(4,1) = \frac{1}{2}\left(d_H^2(1,3) + d_H^2(2,4)\right).$$

Hence by (2) and Lemma 2.1

$$\begin{aligned}
\|x_1 - x_2\|_2^2 &+ \|x_2 - x_3\|_2^2 + \|x_3 - x_4\|_2^2 + \|x_4 - x_1\|_2^2 \\
&\leq d_H^2(1,2) + d_H^2(2,3) + d_H^2(3,4) + d_H^2(4,1) \\
&= \frac{1}{2}\left(d_H^2(1,3) + d_H^2(2,4)\right) \\
&\leq \frac{1}{2}D^2\left(\|x_1 - x_3\|_2^2 + \|x_2 - x_4\|_2^2\right) \\
&\leq \frac{1}{2}D^2\left(\|x_1 - x_2\|_2^2 + \|x_2 - x_3\|_2^2 + \|x_3 - x_4\|_2^2 + \|x_4 - x_1\|_2^2\right)
\end{aligned}$$

which proves that $D \geq \sqrt{2}$. $\square$

A simple observation allows us to see that the previous bound is tight when we embed Q_2 into ℓ_2 with the identity map, i.e. $x_i = i \in \{0,1\}^2$.

Now we try to generalize this idea to prove lower bounds for the n-dimensional cube Q_n. What we want is to find the analog of (1) in higher dimensions. First we can rewrite this inequality as

$$||x_1 - x_3||^2 + ||x_2 - x_4||^2 - ||x_1 - x_2||^2 - ||x_3 - x_4||^2 - ||x_4 - x_1||^2 \leq 0.$$

Note that we can write the previous inequality in the form

$$\sum_{i,j} b_i b_j \|x_i - x_j\|_2^2 \leq 0,$$

where $b_i \in \{-1, 1\}$. For Q_2 what we did is to assign to each x_i a b_i such that $b_1 = b_3 = 1, b_2 = b_4 = -1$. Note here that $b_1 + b_2 + b_3 + b_4 = 0$ and this gives rise to the following observation.

Lemma 2.3. *(General parallelogram inequalities - A Poincaré inequality) For any $x_i \in \mathbb{R}^d, b_i \in \mathbb{R}, i = 1, \ldots, n$ such that $\sum_i^n b_i = 0$ the following is true:*

$$\sum_{i,j} b_i b_j \|x_i - x_j\|_2^2 \leq 0.$$

Proof. Since $\sum_i^n b_i = 0$ we get that $\sum_{i,j} b_i b_j \left(\|x_i\|_2^2 + \|x_j\|_2^2\right) = 0$. Therefore

$$\begin{aligned}
\sum_{i,j} b_i b_j \|x_i - x_j\|_2^2 &= \sum_{i,j} b_i b_j \left(\|x_i\|_2^2 + \|x_j\|_2^2 - 2x_i \cdot x_j\right) \\
&= \sum_{i,j} b_i b_j \left(\|x_i\|_2^2 + \|x_j\|_2^2\right) - 2\sum_{i,j} b_i b_j \left(x_i \cdot x_j\right) \\
&= -2\sum_{i,j} b_i b_j \left(x_i \cdot x_j\right)
\end{aligned}$$

Let now X be a matrix whose i'th row is the vector x_i, and $b^T = (b_1, \ldots, b_n)$. Then $\sum_{i,j} b_i b_j \left(x_i \cdot x_j\right) = b^t X^t X b = \|Xb\|_2^2 > 0$ and the lemma follows.

□

It is interesting that the way we defined the b_i's for Q_2, we have that the matrix $P = (b_i b_j)_{i,j}$ is positive semidefinite (PSD).

$$P = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Indeed, the matrix is symmetric ($P_{i,j} = b_i b_j = b_j b_i = P_{j,i}$) and for all $x \in \mathbb{R}^4$ we have

$$x^t P x \geq 0$$

The reason is that $P = b^t b$, where $b = (b_1, b_2, b_3, b_4)$. Hence $P = x^t b^t b x = (bx)^2 \geq 0$. Moreover it is trivial to see that $P\overrightarrow{1} = 0$. Lemma 2.3 says that for P we have $\sum_{j=1}^4 p_{i,j} \|x_i - x_j\|_2^2 \leq 0$. This can be generalized by the following lemma.

Lemma 2.4. *Let $P = (p_{i,j})$ be a symmetric PSD matrix such that $P\overrightarrow{1} = 0$. Then for any $x_i \in \mathbb{R}^d, i = 1, \ldots, n$ the following is true:*

$$\sum_{i,j} p_{i,j} \|x_i - x_j\|_2^2 \leq 0.$$

Proof. First note that similarly to Lemma 2.1, we can treat x_i's as being real numbers. Since P is symmetric and $P\overrightarrow{1} = 0$ we have $\sum_{j=1}^n P_{i,j} = 0$ and $\sum_{i=1}^n P_{i,j} = 0$, and therefore $\sum_{i,j} p_{i,j} \left(x_i^2 + x_j^2\right) = \sum_{i,j} P_{i,j} \left(x_i^2 + x_j^2\right) = 0$. What we get then is

$$\begin{aligned}\sum_{i,j} p_{i,j}(x_i - x_j)^2 &= \sum_{i,j} p_{i,j}\left(x_i^2 + x_j^2 - 2x_i x_j\right) \\ &= \sum_{i,j} p_{i,j}\left(x_i^2 + x_j^2\right) - 2\sum_{i,j} p_{i,j}\left(x_i x_j\right) \\ &= -2\sum_{i,j} p_{i,j}\left(x_i x_j\right)\end{aligned}$$

Recall here that the matrix P is PSD, and therefore letting $x^t = (x_1, \ldots, x_n)$ we have

$$\sum_{i,j} p_{i,j}\left(x_i x_j\right) = x^t P x \geq 0,$$

completing the proof[1]. □

Let now $c_2(d) = \inf\{t : (X, d) \overset{t}{\hookrightarrow} \ell_2\}$. We are ready to prove the main theorem.

Theorem 2.5.

1. *For every symmetric positive semidefinite matrix P with $P\overrightarrow{1} = 0$ and for every metric d we have*

$$c_2(d) \geq \sqrt{\frac{\sum_{\{i,j\}:p_{i,j}\geq 0} p_{i,j} d^2(i,j)}{\sum_{\{i,j\}:p_{i,j}<0} -p_{i,j} d^2(i,j)}}$$

2. *If $c_2(d) = D$ then there exists a symmetric positive semidefinite matrix $P = (p_{i,j})$ with $P\overrightarrow{1} = 0$ so that*

$$c_2(d) = \sqrt{\frac{\sum_{\{i,j\}:p_{i,j}\geq 0} p_{i,j} d^2(i,j)}{\sum_{\{i,j\}:p_{i,j}<0} -p_{i,j} d^2(i,j)}}$$

Proof. We only prove the first part and postpone the proof of the second part to the next lectures.

By applying Lemma 2.4 we get $\sum_{i,j} p_{i,j}\|x_i - x_j\|_2^2 \leq 0$, which shows that

$$\sum_{\{i,j\}:p_{i,j}\geq 0} p_{i,j}\|x_i - x_j\|_2^2 \leq \sum_{\{i,j\}:p_{i,j}<0} -p_{i,j}\|x_i - x_j\|_2^2. \tag{3}$$

Now assume that the mapping $i \mapsto x_i$, gives $d \overset{D}{\hookrightarrow} \ell_2$. Now without loss of generality we can assume that the embedding is not expanding i.e.

$$\|x_i - x_j\|_2^2 \leq d^2(i,j) \leq D^2\|x_i - x_j\|_2^2, \qquad \forall i, j \tag{4}$$

[1] See Tutorial 2

Then we have

$$\frac{\sum_{\{i,j\}:p_{i,j}\geq 0} p_{i,j} d^2(i,j)}{\sum_{\{i,j\}:p_{i,j}<0} -p_{i,j} d^2(i,j)} \leq \frac{\sum_{\{i,j\}:p_{i,j}\geq 0} p_{i,j} \|x_i - x_j\|_2^2}{\sum_{\{i,j\}:p_{i,j}<0} -p_{i,j} \|x_i - x_j\|_2^2} \times D^2 \leq D^2,$$

where in the first inequality we used (3) and in the second inequality we used (4). □

Recall here that our goal was to generalize the fact that for the cube Q_2 equipped with the Hamming distance d_H, where $c_2(d_H) = \sqrt{2}$. Theorem 2.5 allows us to prove this generalization.

Proposition 2.6. *For the cube Q_n equipped with the Hamming distance d_H we have $c_2(d_H) = \sqrt{n}$.*

Proof. Assume that $(Q_n, d_H) \overset{D}{\hookrightarrow} \ell_2$. First we will prove that $D \geq \sqrt{n}$. Define the matrix $P = \{p_{i,j}\}$ as follows:

$$p_{i,j} = \begin{cases} -1 & d_H(i,j) = 1 \quad \text{(edges)} \\ 1 & d_H(i,j) = n \quad \text{(antipodes)} \\ n-1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

We will show below in Lemma 2.7 that P is PSD. Moreover note that any $i \in \{0,1\}^n$ has only one antipode and n neighbors. That is

$$\sum_{j=1}^{2^n} p_{i,j} = n - 1 + 1 - n = 0, \qquad i = 1, \ldots, 2^n.$$

Hence $P\overrightarrow{1} = 0$ and so we can apply Theorem 2.5. Before this, just note that for any of the 2^n rows i of Q we have

$$\sum_{\{j\}:p_{i,j}\geq 0} p_{i,j} d_H^2(i,j) = (n-1)\cdot 0 + 1 \cdot n^2 = n^2,$$

and that

$$\sum_{\{j\}:p_{i,j}<0} p_{i,j} d_H^2(i,j) = -n.$$

It follows that

$$D \geq \sqrt{\frac{\sum_{\{i,j\}:p_{i,j}\geq 0} p_{i,j} d^2(i,j)}{\sum_{\{i,j\}:p_{i,j}<0} -p_{i,j} d^2(i,j)}} = \sqrt{\frac{2^n n^2}{2^n n}} = \sqrt{n}$$

The previous lower bound can be proven to be tight. It is easy to see that the distortion of the identity mapping from Q_n to $\mathbb{R}^n$ is exactly $\sqrt{n}$ since for $x \in \{0,1\}^n$, by Cauchy-Schwartz inquality we have $\sqrt{n}\|x\|_2 \geq \|x\|_1 \geq \|x\|_2$. □

It only remains to show that the matrix Q as defined in the previous proposition is PSD. Note here that if A is the adjacency matrix of Q_n and P the matrix with $P_{i,j} = 1$ if i and j are antipodes, and 0 otherwise, then $Q = (n-1)I - A + P$.

Lemma 2.7. *The matrix $Q = (n-1)I - A + P$, where A is the adjacency matrix of Q_n and P is the matrix of the antipodes respectively, is PSD.*

Proof. For any vector $x \in \{0,1\}^n$, we consider the vector v_x, such that its y-th coordinate is $(v_x)_y = (-1)^{\langle x,y \rangle}$. We will show that each such v_x is an eigenvector for Q and moreover that it corresponds to a positive eigenvalue.

First let us calculate the y-th coordinate of Av_x.

$$\begin{aligned}
(Av_x)_y &= \sum_{z \in Q_n : d(y,z)=1} (v_x)_z \\
&= \sum_{i=1}^{n} (v_x)_{(y+e_i) \mod 2} \\
&= \sum_{i=1}^{n} (-1)^{\langle x, y+e_i \rangle} \\
&= \sum_{i=1}^{n} (-1)^{\langle x,y \rangle} (-1)^{\langle x, e_i \rangle} \\
&= (-1)^{\langle x,y \rangle} \sum_{i=1}^{n} (-1)^{\langle x, e_i \rangle} \\
&= (v_x)_y \left(n - 2 \sum_{i=1}^{n} x_i \right)
\end{aligned}$$

Similarly, for the y-th coordinate for Pv_x we have

$$(Pv_x)_y = (v_x)_{\overrightarrow{1}-y} = (-1)^{\langle x, \overrightarrow{1}-y \rangle} = (-1)^{-\langle x,y \rangle} (-1)^{\langle x, \overrightarrow{1} \rangle} = (v_x)_y (-1)^{\langle x, \overrightarrow{1} \rangle}$$

Therefore the eigenvalue of $(n-1)I - A + P$ that corresponds to the eigenvector v_x is

$$n - 1 + 2\sum_{i=1}^{n} x_i - n + (-1)^{\langle x, \overrightarrow{1} \rangle} = 2\sum_{i=1}^{n} x_i - 1 + (-1)^{\langle x, \overrightarrow{1} \rangle} \geq 0$$

completing the proof. ∎

References

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