# CSC 2414H (Metric Embeddings) - Assignment 4

Due April 10, 2006

**General rules :** In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade.

1. Prove that it is possible to embed the complete binary tree on $n$ vertices into $\ell_2$ with constant *average* distortion.

2. Give an example of a planar graph that does not embed into the Euclidian plane $\mathbb{R}^2$ with distortion $o(\sqrt{n})$. Hint: take star with $n+1$ nodes for example.

3. Show that every $\ell_2^2$ distance (i.e. $d(x,y) = \|x-y\|_2^2$ and note that this is not necessarily a metric) on $n$ points can embed to a metric with distortion $O(n)$. Prove that sometime $\Omega(n)$ is indeed needed.

4. Recall that the in the structure-theorem there was a set $X = \{x_1, \ldots, x_n\} \subset B(0,1)$ with $d_{ij} = \|x_i - x_j\|^2$ a metric, with constant average distance.

   (a) Show that there are positive constants $\nu, \xi, \rho$ (that do not depend on $n$) so that for every $X$ as above there is a point $z_0 \in B(0,1)$ and a subset $Z \subset X$ so that

   - $\|z - z_0\| \geq \nu$ for every $z \in Z$.
   - $|Z| \geq \xi n$.
   - the average distances of the square Euclidean distances of points in $Z$ is at least $\rho$.

   (b) Use this to give the precise analysis of phase I of SetFind, ie showing that the probability that $S_u$ and $T_u$ are both of linear size. (recall that the analysis presented in class only covered the case that $X$ is in the unit sphere.)

5. Let $X \subseteq S^{m-1}$ be a $2n$ point space, and let $d(x,y) = \|x-y\|^2$ be a metric. Assume further that if $x \in X$ then also $-x \in X$. Prove that there exists subsets $A, B \subseteq X$ for which $A = -B (= \{-x | x \in A\})$, $|A| = |B| = \Omega(n)$ and $d(A,B) \geq \Delta$ for $\Delta = \Omega(1/\sqrt{\log n})$. Hint: apply set find carefully. Why are the conditions similar to the usual setting of the structure theorem?