Due: 2pm on Monday October 1st. **Worth**: 5%

Keep in mind that your proofs will be graded on their structure at least as much as on their content. In other words, when we ask you to prove a statement, it is to find out how well you can write proofs, not because we are interested in knowing whether or not that statement is true. So pay particular attention to writing your proofs properly.

1. [10 marks]

   Use the principle of well ordering to prove that forall $p \in \mathbb{N}$, and for all $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $(m+1)^p = km + 1$. (Proofs using other forms of induction will receive at most 1/2 marks.)

2. [10 marks]

   How many subsets of even size does an arbitrary set of size n contain? Prove your claim.

   For example, the set $\{a, b\}$ has subsets $\{\}$, $\{a, b\}$ of even size and subsets $\{a\}$, $\{b\}$ of odd size; the set $\{a, b, c\}$ has subsets $\{\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ of even size and subsets $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b, c\}$ of odd size.

3. [10 marks]

   Recall that the principle of simple induction can be stated as follows, for any predicate $P(n)$ (in this question, all quantifiers are over $\mathbb{N}$):

   if $\big(P(0)$**, and for all** n**,** $P(n) \rightarrow P(n+1)\big)$ **then for all** n $P(n)$

   To use this, we prove the statement $\big(P(0)$, and for all n, $P(n) \rightarrow P(n+1)\big)$ and then conclude for all n $P(n)$.

   For each statement below, suppose that we have proven that statement; give the value(s) of n for which we can conclude $P(n)$ and justify each answer briefly.

   (a) $P(0)$, and for all $n, P(n) \rightarrow P(n+2)$

   (b) $P(0)$, and for all $n > 0, P(n-1) \rightarrow P(n)$

   (c) $P(0)$, and for all $n > 0, P(n) \rightarrow P(n+1)$

   (d) $P(1)$, and for all $n > 0, P(n) \rightarrow P(n+1)$

   (e) $P(2)$, and for all $n, P(n) \rightarrow P(n+2)$ and for all $n > 1, P(n) \rightarrow P(n-1)$

4. [10 marks]

   A "triangle array" of size k is a tiling of a triangle into similar sub-triangles, with k smaller triangle per side. For example, a triangle array of size 4 is pictured on the right.
   A "walk" in a triangle array is a path that travels from one sub-triangle to one of its neighbours, through a common edge, without ever visiting the same sub-triangle twice. For example, a walk has been started in the picture on the right.
   Prove that for all $k \in \mathbb{N}$, every triangle array of size k contains a walk that visits at least $k^2 - k + 1$ many sub-triangles.