1. [10 marks]

   For a contradiction, suppose $\exists p \in \mathbb{N}, \exists m \in \mathbb{N}, \forall k \in \mathbb{N}, (m+1)^p \neq km+1$. Consider the set $S = \{p \in \mathbb{N} : \exists m \in \mathbb{N}, \forall k \in \mathbb{N}, (m+1)^p \neq km+1\}$. By our assumption, $S \neq \varnothing$ so by the principle of well ordering, there is a smallest element $\hat{p} \in S$. By definition of S, $\exists m \in \mathbb{N}, \forall k \in \mathbb{N}, (m+1)^{\hat{p}} \neq km+1$.

   - $\hat{p} \neq 0$ because for all $m \in \mathbb{N}$, $(m+1)^0 = 1 = 0 \cdot m + 1$, *i.e.*, $\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, (m+1)^0 = km+1$.
   - So $\hat{p} > 0$ and $\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, (m+1)^{\hat{p}-1} = km+1$ (since $\hat{p}$ is the smallest element of S). However, for all $m \in \mathbb{N}$,

   $$\begin{aligned}(m+1)^{\hat{p}} &= (m+1)(m+1)^{\hat{p}-1} \\ &= (m+1)(\hat{k}m+1) \\ &= \hat{k}m^2 + \hat{k}m + m + 1 \\ &= (km+m+1)m+1 \\ &= k'm+1\end{aligned}$$

   where $\hat{k}$ is the natural number whose existence is guaranteed by $\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, (m+1)^{\hat{p}-1} = km+1$. Hence, $\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, (m+1)^{\hat{p}} = km+1$, which contradicts the definition of $\hat{p}$.

   Therefore, it must be the case that S is empty, *i.e.*, the original statement is true.

2. [10 marks]

   Claim: For all $n \geq 1$, every set of size n has exactly 2^{n-1} even-sized subsets.

   Proof: By induction on n.

   Base Case: Let S be a set of size 1, *i.e.*, $S = \{a\}$. Then, S has subsets $\varnothing$ and $\{a\}$, and only one of them has even size: $\varnothing$. Hence, S has exactly $1 = 2^{1-1}$ even-sized subsets.

   Ind. Hyp.: Let $n \geq 1$ and suppose that every set of size n has exactly 2^{n-1} even-sized subsets.

   Ind. Step: Let S be a set of size $n+1$, *i.e.*, $S = \{a_1, \ldots, a_n, a_{n+1}\}$. Then, the even-sized subsets of S that do not contain a_{n+1} are exactly the same as the even-sized subsets of $\{a_1, \ldots, a_n\}$, so there are 2^{n-1} many such subsets by the IH. In addition, the even-sized subsets of S that contain a_{n+1} are exactly the same as the odd-sized subsets of $\{a_1, \ldots, a_n\}$ to which a_{n+1} has been added, so there are 2^{n-1} many such subsets by the IH (since there are 2^n many subsets, as proved in class, and 2^{n-1} many even-sized subsets by the IH). Therefore, S has exactly $2^{n-1} + 2^{n-1} = 2^n = 2^{n+1-1}$ even-sized subsets, as desired.

3. [10 marks]

   (a) $P(n)$ holds for all even $n \in \mathbb{N}$: $P(0)$ holds, $P(2)$ holds because $P(0) \rightarrow P(2)$, $P(4)$ holds because $P(2) \rightarrow P(4)$, ... Also, $P(1) \rightarrow P(3)$, $P(3) \rightarrow P(5)$, ..., but we do not know that $P(n)$ holds for any odd value of n.

   (b) $P(n)$ holds for all $n \in \mathbb{N}$: $P(0)$ holds, $P(1)$ holds because $P(0) \rightarrow P(1)$, $P(2)$ holds because $P(1) \rightarrow P(2)$, ...

   (c) $P(n)$ holds for $n = 0$ only: $P(0)$ holds. Also, $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, ... but we cannot say $P(n)$ holds for any $n > 0$.

   (d) $P(n)$ holds for all $n \geq 1$: $P(1)$ holds, $P(2)$ holds because $P(1) \rightarrow P(2)$, $P(3)$ holds because $P(2) \rightarrow P(3)$, ... However, we do not know whether $P(0)$ holds.

   (e) $P(n)$ holds for all $n \geq 1$: $P(2)$ holds, $P(1)$ holds becasue $P(2) \rightarrow P(1)$, $P(3)$ holds because $P(1) \rightarrow P(3)$, $P(4)$ holds because $P(2) \rightarrow P(4)$, ... However, we do not know whether $P(0)$ holds.

4. [10 marks]

We prove that for all $k \geq 1$, there is a walk that visits $k^2 - k + 1$ sub-triangles and that ends at a corner of the triangle array of size k.

Base Case: For $k = 1$, there is only one sub-triangle in the triangle array, and only one possible "walk" that starts and ends on the sub-triangle (which happens to be a corner). This walk visits exactly $1 = 1^2 - 1 + 1$ sub-triangles.

Ind. Hyp.: Let $k \geq 1$ and suppose that for all triangle arrays of size k, there is a walk that visits at least $k^2 - k + 1$ sub-triangles and that ends at a corner.

Ind. Step:

Consider a triangle array of size $k+1$. This consists of a sub-array of size k above one last row of sub-triangles. By the Ind. Hyp., there is a walk that visits k^2-k+1 sub-triangles and that ends at a corner of the sub-array of size k. Without loss of generality, we can ensure that the last corner of that walk is the rightmost one on the bottom row of the sub-array of size k (by rotating and/or flipping the walk as needed). Then, the walk can be extended to the last row in the "obvious" way, to visit $(k^2 - k + 1) + 2k = (k^2 + 2k + 1) - k - 1 + 1 = (k + 1)^2 - (k + 1) + 1$ sub-triangles and end at a corner. This is illustrated on the right.