1. [10 marks]

   Conjecture: For all networks N, $C(N) = 2L(N)$, where $C(N)$ is the sum of the connectivities of all hubs of N and $L(N)$ is the total number of links of N.

   Predicate: Let $P(n)$ be the statement "for all networks N with n links, $C(N) = 2L(N)$". We prove $\forall n, P(n)$ by induction on n.

   Base Case: Let N be a network with 0 links. Then, the connectivity of each hub is 0 so $C(N) = 0 = 2 \cdot 0 = 2L(N)$.

   Ind. Hyp.: Let $n \in \mathbb{N}$ and suppose that for every network N with n links, $C(N) = 2L(N)$.

   Ind. Step: Let N be a network with $n+1$ links. Let N' be obtained from N by removing a single link (this is always possible because $n+1 \geq 1$). Then, N' contains n links and by the IH, $C(N') = 2n = 2L(N')$. However, $C(N') = C(N) - 2$ because the removal of one link from N reduces the connectivity of both endpoints of that link by 1, and $L(N') = L(N) - 1$ because we've remove a single link. Hence, $C(N) = C(N') + 2 = 2L(N') + 2 = 2(L(N') + 1) = 2L(N)$, as desired.

   Note that it was also possible to prove this by induction on the number of hubs in the network. Obviously, you did not have to give *both* proofs (one is sufficient), but we include both of them here for your reference.

   Predicate: Let $P(n)$ be the statement "for all networks N with n hubs, $C(N) = 2L(N)$". We prove $\forall n, P(n)$ by induction on n. In the proof, we use $C(h)$ to denote the connectivity of a hub h.

   Base Case: Let N be a network with 0 hub. Then, $C(N) = 0 = 2 \cdot 0 = 2L(N)$.

   Ind. Hyp.: Let $n \in \mathbb{N}$ and suppose that for every network N with n hubs, $C(N) = 2L(N)$.

   Ind. Step: Let N be a network with $n+1$ hubs. Let N' be obtained from N by removing a single hub h, along with each link involving h (this is always possible because $n+1 \geq 1$). Then, N' contains n hubs and by the IH, $C(N') = 2L(N')$. However, $C(N') = C(N) - 2C(h)$ because the removal of hub h from N removes $C(h)$ from the connectivity of N twice: once for h and a second time for each hub h was linked with. Also, $L(N') = L(N) - C(h)$ because the removal of hub h from N removes $C(h)$ links. Hence, $C(N) = C(N') + 2C(h) = 2L(N') + 2C(h) = 2(L(N') + C(h)) = 2L(N)$, as desired.

2. [10 marks] First, a little review of modulo arithmetic. For all $n \in \mathbb{Z}$, $n \bmod 11$ denotes the remainder when dividing n by 11 (*i.e.*, $n \bmod 11 = n - 11\lfloor \frac{n}{11} \rfloor$). It can be proved that for all $m, n \in \mathbb{Z}$, $(n \bmod 11) \bmod 11 = n \bmod 11$ and $(m+n) \bmod 11 = (m \bmod 11 + n \bmod 11) \bmod 11$. In particular, for all $n \in \mathbb{N}$, $-n \bmod 11 = (11 - n) \bmod 11 = (11 - n \bmod 11) \bmod 11$. For example, $-26 \bmod 11 = (11 - 26 \bmod 11) \bmod 11 = (11 - 4) \bmod 11 = 7$ and $-33 \bmod 11 = (11 - 33 \bmod 11) \bmod 11 = 11 \bmod 11 = 0$.

   Predicate: Let $P(n)$ be the predicate "$n \bmod 11 = (S_0(n) - S_1(n)) \bmod 11$". We prove $\forall n, P(n)$ by induction on n.

   Base Cases:
   - For $0 \leq n \leq 9$, $n \bmod 11 = (n - 0) \bmod 11 = (S_0(n) - S_1(n)) \bmod 11$.
   - $10 \bmod 11 = (11 - 1) \bmod 11 = -1 \bmod 11 = (0 - 1) \bmod 11 = (S_0(10) - S_1(10)) \bmod 11$.

   Ind. Hyp: Let $n > 10$ and suppose $j \bmod 11 = (S_0(j) - S_1(j)) \bmod 11$ for $0 \leq j < n$.

   Ind. Step: Say $n = d_k d_{k-1} \ldots d_1 d_0$ is the decimal representation of n and let $m = d_k d_{k-1} \ldots d_1$, *i.e.*, $m = \lfloor n/10 \rfloor$, so $n = 10m + d_0$. Then, $0 \leq m < n$ because $n > 10$ so the IH implies $m \bmod 11 = (S_0(m) - S_1(m)) \bmod 11$. Also, $S_1(n) = S_0(m)$ and $S_0(n) = S_1(m) + d_0$ and $d_0 \bmod 11 = d_0$, so we get exactly what we need:

$$\begin{aligned}
n \bmod 11 &= (10m + d_0) \bmod 11\\
&= (11m - m + d_0) \bmod 11\\
&= (-m + d_0) \bmod 11\\
&= (-m \bmod 11 + d_0 \bmod 11) \bmod 11\\
&= (11 - m \bmod 11 + d_0) \bmod 11\\
&= (11 - (S_0(m) - S_1(m)) \bmod 11 + d_0) \bmod 11\\
&= (-(S_0(m) - S_1(m)) + d_0) \bmod 11\\
&= (S_1(m) + d_0 - S_0(m)) \bmod 11\\
&= (S_0(n) - S_1(n)) \bmod 11
\end{aligned}$$

Hence, for all $n \in \mathbb{N}$, $n \bmod 11 = 0$ iff $S_0(n) - S_1(n) \bmod 11 = 0$, *i.e.*, n is divisible by 11 iff $S_1(n) - S_0(n)$ is divisible by 11.

3. [5 marks]

Let $S(n)$ be the number of sequences of n bits in which there are no consecutive 1s.

- $S(0) = 1$ because there is a single sequence of 0 bits (the empty sequence) and it does not contain consecutive 1s.
- $S(1) = 2$ because there are two sequences of 1 bit (0 and 1), neither of which contain consecutive 1s.
- For $n > 1$, $S(n) = S(n-1) + S(n-2)$ because every sequence of n bits in which there are no consecutive 1s either ends with 0 or with 1. There are $S(n-1)$ sequences of n bits that end with 0 and in which there are no consecutive 1s (one for every sequence of $n-1$ bits in which there are no consecutive 1s), and there are $S(n-2)$ sequences of n bits that end with 1 and in which there are no consecutive 1s (one for every sequence of $n-2$ bits in which there are no consecutive 1s), because such sequences must end with last two bits 01 (we know that there are at least two bits because $n > 1$).

4. [15 marks]

- $T(n) = 0$ for all even n, because no full binary tree contains an even number of nodes (proved in textbook).
- $T(1) = 1$, because there is only one full binary tree with 1 node.
- $T(n) = \sum_{i=1,3,\ldots,n-2} T(i) \cdot T(n-1-i)$ for all odd $n > 1$, for the following reasons:
  - Every full binary tree with $n > 1$ nodes consists of a root together with some odd number of nodes $i \geq 1$ in the left subtree, which leaves $n - i - 1$ nodes for the right subtree. Moreover, the full binary trees for one value of i are distinct from those for different values of i, so we account for all possibilities exactly once.
  - For each odd value of $i \geq 1$, there are $T(i)$ distinct subtrees with i nodes (for the left subtree) and for each one of those, there are $T(n-1-i)$ distinct subtrees with $n-1-i$ nodes (for the right subtree). In total, this makes $T(i) \cdot T(n-1-i)$ distinct ways to arrange the nodes in both subtrees.

Proof that $T(n) \geq \frac{2^{(n-1)/2}}{n}$ for all odd $n \geq 1$, by induction.

Base Cases: $T(1) = 1 \geq 1 = \frac{2^{(1-1)/2}}{1}$.
$T(3) = T(1) \cdot T(1) = 1 \geq \frac{2}{3} = \frac{2^{(3-1)/2}}{3}$.

Ind. Hyp.: Let $n > 3$, n odd, and suppose that $T(j) \geq \frac{2^{(j-1)/2}}{j}$ for all odd j such that $1 \leq j < n$.

Ind. Step:

$$
\begin{aligned}
T(n) &= \sum_{i=1,3,\ldots,n-2} T(i) \cdot T(n-i-1) \\
&\geq \sum_{i=1,3,\ldots,n-2} \frac{2^{(i-1)/2}}{i} \cdot \frac{2^{(n-i-1-1)/2}}{n-i-1} \\
&\geq \sum_{i=1,3,\ldots,n-2} \frac{2^{(n-3)/2}}{i(n-i-1)} \qquad \textit{(need $n > 3$ for next step)} \\
&\geq 2^{(n-3)/2} \left(\frac{1}{1(n-1-1)} + \frac{1}{(n-2)(n-(n-2)-1)} + \sum_{i=3,5,\ldots,n-4} \frac{1}{i(n-i-1)} \right) \\
&\geq 2^{(n-3)/2} \frac{2}{n-2} \qquad \textit{(actually strictly greater)} \\
&\geq \frac{2^{(n-1)/2}}{n-2} \\
&\geq \frac{2^{(n-1)/2}}{n}
\end{aligned}
$$