Worth: 8% **Due:** Thursday November 5

For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.

1. Prove or disprove: Whenever L_1 is a decidable language and L_2 is a language different than $\varnothing$ and Σ it holds that $L_1 \leqslant_m L_2$.

   (HINT: The facts that $L_2 \neq \varnothing$ and $L_2 \neq \Sigma^*$ implies the existence of certain things.)

2. For each language L below, state whether L is decidable, undecidable but recognizable, or unrecognizable, then prove your claim.

   (a) $L_1 = \big\{\langle M_1, M_2\rangle : M_1, M_2 \text{ are TMs such that } L(M_1) \subseteq L(M_2)\big\}$

   (b) $L_2 = \big\{\langle M\rangle : M \text{ is a TM such that there is an input on which } M \text{ runs less than 363 steps}\big\}$

3. For any TM M with input alphabet $\{0, 1\}$ and tape alphabet $\{0, 1, \sqcup\}$, define $r(M)$ to be the number of tape cells that are read at least once during the computation of M on input ε (*i.e.*, starting with a blank tape), as long as that number is finite—if not, set $r(M) = 0$. Tape cells are counted only the first time that they are read, so this measures the total amount of tape necessary to carry out the computation of M on input ε, when that amount is finite.

   Next, define $R(n) = \max\big\{r(M) : M \text{ is a TM with } n + 2 \text{ states}\big\}$, *i.e.*, $R(n)$ is the maximum finite number of tape cells that are read at least once by any TM with $n+2$ states during its computation on input ε. Because there are a finite number of TMs with $n+2$ states, the function $R(n)$ is well-defined for each value of n. For example,

   - $R(0) = 0$ because there are only two TMs with 2 states: one that starts in state q_A (the accepting state) and one that starts in state q_R (the rejecting state), neither of which reads any tape cell during its computation.
   - $R(1) = 2$ because there is a TM M with 3 states that reads 2 different tape cells when it is started on a blank tape, but all other TMs with 3 states either read no more than 2 tape cells or read an infinite number of tape cells when they are started on a blank tape—you should verify this for yourself. M has state set $\{q_1, q_A, q_R\}$, initial state q_1, and M's transition function is defined as follows: $d(q_1, \sqcup) = (q_1, 0, L)$, $d(q_1, 0) = (q_1, 1, R)$, $d(q_1, 1) = (q_A, \sqcup, R)$,

   Prove that $R(n)$ is *not* a computable function.

   (HINT: There are two kinds of infinite loops that TMs can get stuck into: loops that visit more and more tape cells, and loops that use only a finite amount of tape. One of these kinds of loops can be detected.)

4. Can you use Rice's Theorem to prove the undecidability of the language $L = \big\{\langle M\rangle : M \text{ is a TM such that } |L(M)| = 715\big\}$?