1. Let $E = \{ x : (x-c)^t Q^{-1} (x-c) \leq 1 \}$ be an ellipsoid enclosing a (nonempty) polytope $P$ with smallest possible volume. Is it possible that $c$ (the centre of $E$) does not belong to $P$? Either give an example of such a polytope and its smallest enclosing ellipsoid or prove that it is impossible.

2. Recall that one of the steps of the ellipsoid algorithm was to define a large enough cube $B = [-K, K]^n$ such that if $P = \{ x : Ax < b \}$ is nonempty then also $P \cap B$ is nonempty and moreover there is a lower bound on its volume $2^{-\Omega(L)}$ where $L$ is the length of the description of $A$. Remember that in class such a construction and proof were given for the case where we assume that $P$ is bounded. Your goal is to prove the existence of such a box for $P$ that is not necessarily bounded.

3. Consider the following optimization problem: Let $n$ be even and let $c$ be a positive vector in $\mathbb{R}^n$, find

$$\min \{ \langle c, x \rangle \mid x \geq 0 \text{ and } \sum_{i \in S} x_i \geq 1 \ \forall S \subset \{1, 2, \ldots, n\} \text{ of size } n/2 \}. $$

This is an LP with exponentially (in $n$) many constraints.

(a) Show that the Ellipsoid algorithm can be used to get a poly($n$)-time algorithm for the problem.

(b) Show a simpler ($O(n \log n)$ time) algorithm for the problem. Hint: It may be useful to first find a simpler LP for the problem. You can then use facts from the theory of polynomial to get a simple algorithm that doesn’t actually solve an LP (obviously such an algorithm is too costly).

4. Recall that in an interior point algorithm we require a potential function that takes into account the objective function as well as the distance from the boundary of the feasible set. In Ye’s Primal-dual interior point algorithm we have $x$ the primal variables, and $y$ and $s$ the dual variables, where $s$ is the slack variables. We have also seen that if we have a primal-dual pair of solutions then $x \cdot s$ is an upper bound on the gap between the current value of the objective function and its optimum. Our goal was then to minimize $x \cdot s$ (and recall $x, s \geq 0$ always.)

The potential function that we discussed was

$$G(x, s) = (n + \sqrt{n}) \ln(x \cdot s) - \sum_{i=1}^{n} \ln(x_i s_i)$$
and we showed the critical property that we can upper bound $x \cdot s$ in terms of $G(x, s)$. Explain what would go wrong if instead of $G$ we will take

$$H(x, s) = n \ln(x \cdot s) - \sum_{i=1}^{n} \ln(x_i s_i).$$

5. (a) Let $A$ be a totally unimodular matrix and let $l_1, l_2, u_1$ and $u_2$ be integral vectors. Show that the LP relaxation of the following Integer Program is exact (i.e. all its vertices are integral).

\[
\begin{align*}
\text{min} & \quad \langle x, c \rangle \\
\text{s.t.} & \quad l_1 \leq Ax \leq u_1 \\
& \quad l_2 \leq x \leq u_2 \\
& \quad x \in \mathbb{Z}^m
\end{align*}
\]

(b) Let $A$ be an $m \times n$ integer matrix of rank $m$. Show that $A$ is unimodular (that is, the determinant of all $m \times m$ submatrices of $A$ is $-1, 0$ or $1$) if and only if for every integral vector $b$ the vertices of the polyhedron

$$P = \{x \mid x \geq 0, Ax = b\}$$

are all integral.