CSC 2402H - Assignment 1

Due Oct 20, 2009

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade.

1. Let G be an undirected simple graph on n vertices $\{1, 2, \ldots n\}$. We define the Tutte matrix of G, as the n by n matrix $A = A(G)$ with $A_{ij} = 0$ if $ij \notin E(G)$, $A_{ij} = x_{ij}$ if $ij \in E$ and $i < j$ and $A_{ij} = -x_{ji}$ if $ij \in E$ and $i > j$.

   Show that there exists a perfect matching in G if and only if the determinant of A(G) (which is a polynomial in the x_{ij}s) is not the zero polynomial.

2. **(a) minimizing an objective functions which is a (finite) maximum** Consider the following problem.

   $$\begin{aligned} & \min_x \max_{i \in I} x_i \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

   Where $x = (x_1, \ldots, x_n)^t$, and I is a fixed subset of $\{1, \ldots, n\}$. Show how to express the problem as an LP. You may add variables and linear constraints as long as there is an easy way to get a solution to the original problem from the newly formulated problem.

   (b) Let $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be a set of n points in $\mathbb{R}^2$ so that $x_i \neq x_j$ for $i \neq j$. Your goal is to find a line $ax + b$ so that the maximal *vertical* distance between the line and the points is minimized (the vertical distance of (x_i, y_i) to the line is $|ax_i + b - y_i|$). Write an LP for the problem.

3. **(a) size of BFS** In class we stated without proof that the length of numbers involved in a BFS are polynomially related to the input size. More precisely, consider an LP in standard form (which means that the constraints are $Ax = b, x \geq 0$) with integer coefficients (the entries of A and b), n variables and m equalities, and let $L = mn + \lceil \log |p| \rceil$, where p is the product of the nonzero coefficients in A,b and c. Then all coordinates of any BFS are rational numbers with both the absolute value and denominator bounded by 2^L. Prove it. Hint : Let B be a square nonsingular matrix with small coefficients, how can the coordinates of $B^{-1}b$ be bounded?

   (b) feasibility is hard Use (a) to show that the Feasibility problem is "as hard" as Optimality problem in LP. In other words, one can use a subroutine which returns a feasible point for an LP in standard form or reports the problem is infeasible, in order to solve the optimality problem for an LP in standard form.

4. **BFS in nonstandard form.** Analogously to the definition of BFS for LP in standard form, we may define the BFSs for

$$P = \{x : Ax \leq b\}$$

as the points y for which there are n linearly independent inequalities satisfied as equalities, where n is the number of variables. In class we mentioned that a point x in a polytope Q is called *extreme* if whenever $x = \lambda y + (1 - \lambda)z$ for some $y, z \in Q$, and $0 \leq \lambda \leq 1$, we must have $x = y = z$. In other words x is not a convex combination of other points in Q. Show that in this setting a point is BFS iff it is extreme.