CSC 2411H - Assignment 3

Due Nov 27, 2009

1. (a) Let A be a totally unimodular matrix and let L_1, L_2, U_1 and U_2 be integral vectors. Show that the LP relaxation of the following Integer Program is *exact* (i.e., all its vertices are integral).

$$\begin{array}{rl} & \min x \cdot c \\ \text{subject to} & L_1 \leq Ax \leq U_1 \\ & L_2 \leq x \leq U_2 \\ & x \in \mathbb{Z}^m \end{array} .$$

   (b) Let A be an $m \times n$ integer matrix of rank m. Show that A is unimodular (that is, the determinant of all $m \times m$ submatrices of A is $-1, 0$ or 1) if and only if it holds for very integral vector b that all vertices of the polyhedron $P = \{x \mid x \geq 0, Ax = b\}$ are integral.

2. For a graph G on n vertices, we say that a permutation σ of its vertices is an automorphism if for all $i, j \in V(G)$

$$\sigma(i)\sigma(j) \in E(G) \text{ iff } ij \in E(G).$$

The set of all automorphism of a graph $\mathrm{Aut}(G)$ is a group under the composition operation. (verify this to yourself).

Our goal is to relate symmetry conditions of an optimal solution of an SDP of a graph problem with automorphisms of the graph. We will demonstrate this using the MaxCut SDP relaxation (GW). Recall the relaxation:

$$\begin{array}{lll} \max \frac{1}{2} \sum_{ij \in E(G)} (1 - \langle v_i, v_j \rangle) & & \\ \text{subject to} \quad v_i \cdot v_i = 1 & \text{for all } i \in V(G) & . \\ \qquad v_i \in \mathbb{R}^n & \text{for all } i \in V(G) & \end{array}$$

   (a) For an $n \times n$ matrix A and for a permutation σ on the vertices define a matrix $A^{(\sigma)}$ as follows

$$A^{(\sigma)}_{ij} = A_{\sigma(i)\sigma(j)}$$

   Assuming X is a feasible solution to the SDP, what can be said about $\sum_{\sigma \in \mathrm{Aut}(G)} X^{(\sigma)}$?

   (b) Let $\sigma \in \mathrm{Aut}(G)$. Use the first part to show that there is an optimal solution for the SDP, Y that is symmetric with respect to σ. In other words, $Y_{ij} = Y_{\sigma(i)\sigma(j)}$ for all $i, j \in V(G)$.

(c) Use the above to find the integrality gap for the GW relaxation for C_3,C_4 and C_5, the cycles on $3, 4, 5$ vercies respectively.

3. We are given a set $S = \{x_1, x_2, \ldots, x_n\}$ of n items and a set T of m triplets $T \subseteq S \times S \times S$. Each triplet consists of three distinct items. A total ordering (permutation) of S, $x_{\pi_1} < x_{\pi_2} < \ldots < x_{\pi_n}$ *satisfies* a triplet $(x_i, x_j, x_k) \in T$ if x_j occurs between x_i and x_k in the ordering, i.e., if either $x_i < x_j < x_k$ or $x_k < x_j < x_i$ holds . The problem is to find a total ordering that maximizes the number of satisfied triplets. It can be shown that this is an **NP**-hard problem.

   (a) Show that a random ordering (i.e., a permutation chosen uniformly at random among all possible permutations) will satisfy in expectation one third of all triplets in T.

   Our goal will be to improve the approximation factor to one half.

   (b) Assume that you are provided with an instance that is satisfiable, namely that all m triplets can be satisfied simultaneously. (It can be shown that it is still hard to find satisfying ordering in this limited class of instances). Show that an instance is feasible iff the following quadratic program in variables $p_i \in \mathbb{R}$, $i = 1, \ldots, n$, has a solution:

$$(p_i - p_j)^2 \geq 1 \text{ for all } i, j,$$

$$(p_i - p_j)(p_k - p_j) \leq 0 \text{ for all } (x_i, x_j, x_k) \in T.$$

   (c) Obtain the vector programming relaxation of this quadratic program.

   (d) Give an instance where the above vector program is satisfiable but the instance itself is not satisfiable.

   (e) Let us assume that $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ are vectors that solve the vector program. Now select r uniformly at random on the unit sphere S^{n-1}. Consider the random ordering obtained by sorting $\langle r, v_i \rangle$. Show that, in expectation, this random ordering satisfies at least half of the constraints on T. **Hint:** What is the probability that a single triplet is satisfied? What is the angle between $v_i - v_j$ and $v_k - v_j$?