Robust algorithms for Max Independent Set on Minor-free graphs based on the Sherali-Adams Hierarchy

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Abstract. This work provides a Linear Programming-based Polynomial Time Approximation Scheme (PTAS) for two classical NP-hard problems on graphs when the input graph is guaranteed to be planar, or more generally Minor Free. The algorithm applies a sufficiently large number (some function of $1/\epsilon$ when $1+\epsilon$ approximation is required) of rounds of the so-called Sherali-Adams Lift-and-Project system. needed to obtain a $(1+\epsilon)$-approximation, where f is some function that depends only on the graph that should be avoided as a minor. The problem we discuss are the well-studied problems, the Max Independent Set and Min Vertex Cover problems. An curious fact we expose is that in the world of minor-free graph, the Min Vertex Cover is harder in some sense than the Max Independent Set.
Our main result shows how to get a PTAS for Max Independent Set in the more general "noisy setting" in which input graphs are not assumed to be planar/minor-free, but only close to being so. In this setting we bound integrality gaps by $1+\epsilon$, which in turn provides a $1+\epsilon$ approximation of the optimum value; however we don't know how to actually find a solution with this approximation guarantee. While there are known combinatorial algorithms for the non-noisy setting of the above graph problems, we know of no previous approximation algorithms in the noisy setting. Further, we give evidence that current combinatorial techniques will fail to generalize to this noisy setting.

1 Introduction

A common way to handle NP-hard problems is to design approximation algorithms for them. Often, even a good approximation cannot be achieved if one is concerned with the standard worst-case analysis. For example, it is NP-hard not only to solve Max Independent Set but also to approximate it to within factor of $|V|^\delta$ for any $\delta < 1$ unless NP=ZPP [18]. However, we may be able to to compute good approximations for some classes of inputs. Examples for such classes in the context of graph problem could be graphs with bounded degree, sparse graphs, dense graphs, perfect graphs, etc. In some cases a certain restriction on

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the input renders a problem trivial, such as the case of MAX CLIQUE restricted to bounded-degree graphs; in others, such as SPARSEST CUT on bounded degree graphs are still very hard to approximate. More interesting examples are the semidefinite-programming based algorithm for colouring of perfect graphs [17], or the classical Polynomial Time Approximation Scheme (PTAS) by Arora for Euclidean TSP [2].

In this paper we present algorithms based on Linear Programming (LP), which give rise to a PTAS for the problems of MAX INDEPENDENT SET and MIN VERTEX COVER on minor-free graphs, and in particular on planar graphs.

We first explain how Linear Programming approach may lead to a PTAS, namely algorithms that for each $\epsilon > 0$ give approximation of $1+\epsilon$ and run in time polynomial in the size of the graph and may depend on ϵ. One can think of this as a sequence of algorithms which give approximation factor that approach 1. To come up with such a sequence using an LP, it is natural to consider a sequence of LP formulations rather than a fixed one. Systematic methods that give rise to such sequences are so-called Lift-and-Project methods. Here, the original LP is tightened repeatedly r times (or levels/rounds). When this process is repeated for $r = n$ times, the obtained LP is equivalent to the original Integer Program, and hence solving it will give the exact solution to the original problem, however the running time of such an algorithm will not be polynomial in general. More specifically, starting from a poly-size LP it takes $n^{O(r)}$ to optimize over the level r tightening. In order to obtain a PTAS using the above paradigm, one should show that $\lim \eta_r = 1$ where η_r is the approximation guaranteed by the LP after r rounds of applying the Lift-and-Project operator.

Different variants of Lift and Project methods exist, and in this work we show that the one due to Sherali-Adams satisfies the condition above with respect to some classical graph optimization problems on planar graphs and their generalization to minor-free graphs. To the best of our knowledge there is only one example for PTAS that is obtained by Lift-and-Project systems due to Fernández de la Vega and Kenyon-Mathieu [15] who have provided a PTAS for MAX CUT in dense graphs using the Sherali-Adams hierarchy.

We further consider the setting where the input graphs are noisy, in the sense that they are obtained by applying some bounded number of changes to graphs in the special classes considered above. We show that in this setting the LP-based approach is still effective: we can bound the integrality gap by $1+\epsilon$ when $O(1/\epsilon)$ rounds of Sherali-Adams are applied. It is important to note that while the integrality gap is well-bounded by our method (whence the method well-approximates the optimal value), we don't know how to translate this guarantee to a rounding procedure or to any other method that will obtain a solution approximating the optimum. There aren't many examples in the literature where a bound on the integrality gap is known but no integral solution is presented to achieve the bound, and we note [14] as one example of such a scenario.

Previous Work: Tree graphs, bipartite graphs, small tree-width , outerplanar and planar graphs all have been well-studied in the context of restrictions on the type of input of NP-hard problems. Specifically for our problem, algorithms for planar graphs were studied by Baker [3] who gave a PTAS with running time $O(f(\epsilon)n \log n)$ for Max Independent Set and Min Vertex Cover on planar graphs. For the minor-free case, The work of DeVos et al.[12] opened the way for algorithms in minor-free graph partitioning, as they provided (proof of existence of) a decomposition of the graph to simple parts. Following their work, there were a series of algorithms for minor-free graphs which were mostly nonconstructive [3] such as [16]. However, later in a work of Demaine et al.[13] it was shown that the decomposition can be done in polynomial time which makes those algorithms constructive. We note that our approach is in general inferior to the combinatorial approach in [3] in terms of running time as the time complexity of optimizing in the r-th level of the Sherali-Adams Hierarchy is $n^O(r)$ which means that our algorithm run in time $O(n^{\frac{1}{\epsilon}}$.

In contrast to the above work, no algorithms are known for the noisy setting. In fact, in Section 5 we give evidence that current combinatorial approaches or modification of them are bound to fail.

In the context of PTAS which are LP-based not many examples are known, and we mention two here. In [4], Bienstock shows that a Linear Programming of size polynomial in $1/\epsilon$ and in n to approxmate upto $1 + \epsilon$ the knapsack problem on n items. As in our case, this is an LP-based analogue to an existing combinatorial algorithm, the well known PTAS for Knapsack by Lawler [20]. A second example is due to Avis and Unemoto [11] who show that for dense graphs linear programming relaxations of max cut approximate the optimal solution upto $1 + \epsilon$, where the size of the LP is again polynomial in $1/\epsilon$ and in n. Unlike the current work, however, the LPs in these results are not obtained through the lift-and-project method, but rather they are found in a way customized to the problem. (In fact, for the first result above, even the choice of variables to be used is not obvious.)

Techniques: An essential ingredient in our work is a result by Bienstock and Ozbay [10]. Consider a graph G that has tree-width k, and consider the standard LP relaxation of Max Independent Problem on G. It is shown in [10] that the application of the level k Sherali-Adams (SA) operator gives an *exact* solution to the problem. In other words, the relaxed and integral optimal solutions are the same. The graph-theoretic component of our results uses the theorem of DeVos et. al. [12] mentioned above. The theorem shows that for every positive integer j there is a partition of the vertices of a minor-free graph into j parts so that the removal of *any* of them leaves components of tree-width at most $k(j)$, where $k(j)$ depends only on j and on the minor and not on n. In the special case of planar graphs this decomposition theorem is almost straightforward, with $k(j) = j$. Our approach essentially uses the following simple schema: (i) apply the level-$k(j)$ SA operator, where $j \sim 1/\epsilon$. (ii) bound the *integrality gap* obtained by $1 + \epsilon$. This is made possible by separately bounding the contribution of the

[3] this means that for every H there is an algorithm for the H-minor-free case, but there was no uniform algorithm, that given H and an H-minor-free graph provides the required approximation.

solution on the different parts relative to the corresponding integral solution. Notice that ensuring small integrality gap gives an approximation of the *value* of the optimum and that in order to provide algorithms that actually supply good approximated *solutions* we need to know the decomposition and round the fractional solution according to this decomposition. We later elaborate on this interesting aspect of our technique.

The rest of the paper is organized as follows. In Section 2 we give the relevant graph theoretical definitions as well as the description of the Sherali-Adams Hierarchy. In Section 3 we deal with the Max Independent Set problem. We first show how to get a PTAS for the simpler case of planar graphs, and then extend to family of minor-free graphs. In Section 4 we deal with the approximation of Min Vertex Cover. We show a general lemma that says that under sufficient conditions it is possible to import results about integrality gaps for certain LPs for the problem of Max Independent Set into ones about Min Vertex Cover. Last, we consider the case of graphs which are "noisy versions" of planar or minor-free graphs.. We show that unlike combinatorial approaches, our algorithms can extend to this case. More specifically, we show a PTAS for the *value* of the maximum independent set in noisy planar graphs and, more generally, in noisy H-minor-free graphs.

2 Preliminaries

The tree-width of a graph A tree decomposition of the graph G is a pair (T, X) such that

1. T is tree;
2. for every vertex $v \in G$ there is a tree t_v;
3. $X = \{t_v : V(t_v) \subseteq T\}$ such that each t_v is a subtree of T;
4. and for any edge $e(u, v)$ in $E(G)$, we have $t_v \cap t_u \neq \emptyset$.

We say that a graph G has tree-width k if there exists a tree decomposition of G such that the intersection of every $k + 2$ of t_v's is empty.

The Sherali-Adams Hierarchy Sherali-Adams is a system that given an LP relaxation produces a tightened LP, that will eventually produce a program that is equivalent to the Integer Program describing the problem. More specifically, given a LP relaxation of some $\{0, 1\}$ integer-program on n variables and a parameter r, the Sherali-Adams lifting of the LP in the rth level an LP that is strictly stronger than the original LP and requires $n^{O(r)}$ time to optimize over. When $r = n$, the generated LP is equivalent to the integer program, hence its solution solves the original problem exactly. While this is not essential for the purpose of the current paper, we give below a full description of the system.

For every two disjoint sets of variables I and J such that $|I \cup J| \leq k$, we have a variable $w[I, J]$. This variable represents $\prod_{i \in I} x_i \prod_{j \in J}(1 - x_j)$ in an integer solution, and in particular, an original variable of the LP is associated with $w[\{i\}, \emptyset]$. The system imposes all possible linear conditions on this set of variable that can be derived by (i) the original inequalities of the LP, and (ii) by the relations of the above products amongst themselves. The inequalities of type (i) that we get are derived by every LP inequality For the first type, we obtain the inequality

$$\sum_{j \notin J} a_j w[I \ \cup \{j\}, J] \geq b \cdot w[I, J]. \tag{1}$$

for every LP inequality $\sum_i a_i x_i \geq b$ and every I, J as above.

For type (ii) the following inequalities are obtained.

$$w[\emptyset, \emptyset] = 1 \tag{2}$$

$$0 \leq w[I \cup \{j\}, J] \leq w[I, J] \text{ for } j \notin (I \cup J) \tag{3}$$

$$0 \leq w[I, J \cup \{j\}] \leq w[I, J] \text{ for } j \notin (I \cup J) \tag{4}$$

$$w[I, J] = w[I \ \cup \{j\}, J] + w[I, J \ \cup \{j\}] \tag{5}$$

The obtained linear program "projects back" to the original set of variables, namely considers $w[\{i\}, \emptyset]$. We shall denote by $\mathrm{SA}^{(\mathrm{t})}(\mathrm{G})$ the polytope of all solutions of the t-th level of the Sherali-Adams Hierarchy (this is the extension of the notion of the polytope associated with an LP relaxation).

Noisy Graphs Consider a class of graphs. Then a *noisy version* of a graph from the class is simply a perturbation applied to it. We adopt a standard notion of distance to quantify this: the distance between two graph is the minimum number of edges or vertices that should be added or removed from one of the graphs to become isomorphic to the other graph. We extend this notion to distance between a graph G and a family of graphs in the standard way, namely as the minimum distance of G over all the graphs in the family. Notice that when the family is monotone, that is closed under edge removal, as is the case with the families we consider, the distance is simply the number of edges needed to be removed from the graph in order for it to be in the family. It is important not to confuse the notion of "noise" here, which is deterministic, with the notion of noise used to describe random perturbation of objects, and the result we supply are stronger than corresponding results in the random model.

3 A PTAS for Max Independent Set

In the Max Independent Set problem the input is a graph and the output is an independent set, namely a set of maximum size of vertices that share no edges. This is a classical NP-hard problem which is notoriously hard to approximate. Let n be the number of vertices, then it is NP-hard to approximate the problem to within factor of $n^{1-\epsilon}$ [18]. In other words, in the worst case setting not much can be done. This motivates looking at special classes of inputs.

3.1 Planar Graph Case

While the Max Independent Set problem is still NP-hard for planar graphs, the problem of approximating the solution is quite a bit different. Indeed, any four colouring of a planar graph gives rise to an independent set of size at least $n/4$, and hence 4-approximation algorithm. The next natural is whether a polynomial time algorithm exists that approximate the optimum to within $1+\epsilon$ and what is the dependency in ϵ.

The standard Linear Programming relaxation for the problem is:

$$\boxed{\begin{array}{ll} \text{maximize:} & \sum_{v \in G} x_v \\ \text{for } uv \in E(G)) & x_v + x_u \leq 1 \\ \text{for } u \in V(G) & 0 \leq x_u \leq 1 \end{array}} \tag{6}$$

Notice that this LP is quite weak as the all 1/2 solution is always a feasible solution. For graphs with sublinear independent sets this LP is therefore quite useless as it is. However, it is not hard to show that for planar graphs the integrality gap of the LP above cannot be larger than 2. Our goal now is to show that by using higher level of the Sherali-Adams hierarchy much better approximations can be obtained.

Let G be the input graph and $\alpha(G)$ be the size of the largest independent set of G. Furthermore, let y be the projection of optimal solution of the level k SA operator applied to LP (6) onto the singleton variables. For a set of vertices S we define $y(S)$ as $\sum_{u \in S} y_u$, and $y'(S)$ as $\sum_{u \in S} y_u - y_u^2$. Abusing notation, when M is a graph, we may write $y(M)$ instead of $y(V(M))$.

Fix an embedding of a planar graph G into the plane. Graph G is m-outerplanar for some $m > 0$. The vertices of the graph can be partitioned into m sets $V_1, V_2 \ldots V_m$, where V_1 is the set of vertices in the boundary of the outerface, V_2 is the set of vertices in the boundary of the outerface after V_1 is removed and so on. Note that, if $u \in V_i$ and $w \in V_j$ are adjacent then $|i-j| \leq 1$.

We now wish to remove some of the V_i from the graph so that (i) the remaining graph is k-outerplanar, and (ii) the weight of the removed set in the optimal SA solution is small. Let

$$B(i) = \bigcup_{j = i (\bmod k+1)} V_j.$$

For every value of k this partitions $V(G)$ into $(k+1)$-outerplanar sets. Note that after removing the vertices in $B(i)$, the resulting graph is k-outerplanar. We now consider an index j for which $y'(B(j)) \leq y'(G)/(k+1)$ and denote $B(j)$ by W.

Let G_i be the subgraph of G induced on $V_i = \{v \ : \ v \in V_l, ik + j \leq l \leq (i+1)k + j\}$. Notice that every edge or vertex of G appear in one or two of the G_i, and those vertices not in W appear in precisely one of the G_i. A key observation we need is that applying Sherali-Adams on G and then projecting onto V_i (more precisely, projecting onto all subsets of size at most t in V_i) is a solution in $\mathrm{SA^{(t)}(G_i)}$. This follows from the fact that the LP associated with G is stronger than the one associated with the subgraph G_i (on all common variables) and the same extends to the Sherali-Adams hierarchies. Therefore using [10] we can deduce that the projection of y onto the singleton sets in V_i is a convex combination of integral solutions, namely independent sets of G_i.

Let ρ_i be the corresponding distribution of independent sets for G_i and consider the following experiment (or random rounding): pick a set S_i according to ρ_i, independently for each i. We say that a vertex v is *chosen* if it is in S_i whenever $v \in G_i$. (Notice that for $v \notin W$, v belongs to a unique G_i and the condition is simply that $v \in S_i$, but for $v \in W$, v may belong to both G_i and G_j in which case it is chosen only when $v \in S_i \cap S_j$.) Denote by S the set of chosen vertices. We claim that S is an independent set. Indeed, every edge belongs entirely to some G_i, two neighbours in G_i cannot both be in the independent set S_i, and so they cannot both be chosen.

Since the marginals of ρ_i on $v \in G_i$ is y_v, we get that for vertices $v \notin W$

$$\Pr[v \in S] = y_v$$

and for vertices $v \in W$

$$\Pr[v \in S] \geq y_v^2.$$

From the above conditions we can conclude that

$$\mathsf{E}(|S|) \geq \sum_{v \notin W} y_v + \sum_{v \in W} y_v^2 = \sum_{v} y_v - \sum_{v \in W} (y_v - y_v^2) = y(G) - y'(W)$$

Now, it is easy to see that $y'(G) \leq \frac{3y(G)}{4}$. It is shown in [5] that a k-outerplanar graph has tree-width at most $3k-1$, therefore in the $3k-1$ level of Sherali-Adams y will be integral on any subgraph of tree-width at most k. We can finish off with the required bound

$$IS(G) \geq \mathsf{E}(|S|) \geq y(G) - \frac{1}{k} y'(G) \geq y(G) - \frac{1}{k}\left(\frac{3y(G)}{4}\right) = \left(1 - \frac{3}{4k}\right) y(G)$$

and get

Theorem 1. *Let G be a planar graph. Then $\alpha(G)$ is at least $1 - \frac{3}{4k}$ times the solution of level $3k-1$ Sherali-Adams operator applied on the standard LP for* Max Independent Set *(LP (6)). Further, the above algorithm gives rise to a rounding procedure that actually finds an independent set that is at least $(1 - \frac{3}{4k})\alpha(G)$*

3.2 Extending to Minor-free Graphs

Consider a fixed graph H and consider graphs G which are H-minor-free, namely, they don't contain H as a minor[4] Notice that planar graphs are a special case as they do not contain K_5 (or alternatively, $K_{3,3}$ as a minor. As with the case of

[4] A graph G contains H as a minor if H can be obtained from G by applying a sequence of edge/(isolated)vertex removal and edge contraction.

planar graphs, the special property of a minor-free which is utilized in algorithms is the fact that it can be decomposed into simple components when some limited part of it is removed. As with the case of planar graph, we would like “simple” to stand for small tree width. A recent theorem due to DeVos et al. gives precisely that.

Theorem 2. *(DeVos et. al [12]) For every graph H and integer $j \geq 1$ there exist constants $k_V = k_V(H, j)$ and $k_E = k_E(H, j)$ such that the vertices of every graph G with no H-minor can be partitioned into $j+1$ parts such that the union of every j of them has tree-width at most k_V. In addition, the edges of G can be partitioned into $j+1$ parts such that the union of every j of them has tree-width at most k_E.*

The above theorem is crucial in the algorithm we present. It is worth noting that for the special case of planar graphs we may take k_V to be as small as $O(j)$.

Theorem 3. *For every H and $\epsilon > 0$ there exists a constant $c = c(\epsilon, H)$ such that for every graph G with no H-minor, the integrality gap of the level-c Sherali-Adams operator of LP (6) is at most $1 + \epsilon$.*

Proof. (sketch) Let $c = k_V(H, \lceil 1/\epsilon \rceil)$, we claim that applying level c SA operator is sufficient to derive $1 + \epsilon$ bound on integrality gap. For any subset of vertices we define $y(S) = \sum_{v \in S} y_v$. Using the result from [10]. We know that for any $S \subseteq G$ with tree-width less than or equal to c, we have $y(S) \leq \alpha(S)$. Now if we take the partitioning of vertices into $V_1, \ldots, V_{j+1}$ according to Theorem 2, and remove the partition with minimum $y(V_i)$ from G the rest of the graph must have tree-width at most c, and furthermore we have

$$y(G \setminus V_i) \geq \frac{j}{j+1} y(G),$$

and we bound the integrality gap

$$y(G)/\alpha(G) \leq (1 + 1/j)y(G \setminus V_i)/\alpha(G) = (1 + 1/j)\alpha(G \backslash V_i)/\alpha(G) \leq 1 + 1/j.$$

4 Vertex Cover

A *vertex cover* for a graph G is a subset of the vertices touching all edges. The Min Vertex Cover problem is to find a minimal vertex cover for a graph. For a graph G we denote the minimum vertex cover by $\nu(G)$.

The purpose of this section is to show how to get a SA-based PTAS for Min Vertex Cover on minor-free graphs from a similar PTAS for Max Independent Set. Generally speaking, Min Vertex Cover is easier problem to approximate than its complement, Max Independent Set, and it can be easily approximated by a factor of 2. Notice that an exact algorithm for one problem can be easily converted into an exact algorithm for the other problem. Similarly, the quality of the *additive* approximation to the problems is still the same. It is well known, however, that for the standard measure of approximation namely multiplicative approximation, the approximation quality of the problems may differ dramatically. The most common scenario exhibiting the above difference are graphs with independent sets of size at most $o(n)$ and vertex covers of size at least $n - o(n)$. For the purpose of this section, though, we are interested in understanding the opposite scenario where the size of some vertex covers is $o(n)$; this is since in these such graphs (the compliment of) a $1 + \epsilon$ approximation of MAX INDEPENDENT SET may provide a very poor approximation for MIN VERTEX COVER. Now, there is a standard trick that reduces any instance of MIN VERTEX COVER into one where the optimal solution is of size at least half the graph. This trick simply finds an optimal solution for the standard LP, and removes the vertices who get value 0 in the solution. What we do next avoids the trick. The advantage of having a direct claim about the integrality gap of *any* graph, rather than using it as a subroutine, is that it allows for argument that involves projection of a solution onto smaller subgraphs. Examples of this sort was shown in Section 3, and a more interesting one will be supplied later in Section 5 in the context of noisy graphs.

The LP for MIN VERTEX COVER is formulated below.

$$\begin{array}{|cc|}\hline \text{minimize:} & \sum_{v \in G} x_v \\ \text{for } uv \in E(G)) & x_v + x_u \geq 1 \\ \text{for } u \in V(G) & x_u \geq 0 \\ \hline\end{array} \qquad (7)$$

The idea behind getting a generic statement allowing us to move from MAX INDEPENDENT SET to MIN VERTEX COVER is quite simple. In fact it uses similar reasoning (even if in a more subtle way) to the "standard trick" described above. We split the graph into two parts, one that "behaves integrally" on which no error is incurred, and the other on which the maximum independent set is smaller than the minimum vertex cover, and then combine the two parts. This split is achieved by looking at the optimal solution of the standard LP to MAX INDEPENDENT SET. We start by defining a property of LP relaxations for MAX INDEPENDENT SET.

Downward property: We say that an LP relaxation for MAX INDEPENDENT SET has the *downward property* if its solution y satisfies that for any $S \subseteq V(G)$, $y(S) \leq (1 + \epsilon)\alpha(G')$, where G' is subgraph of G induced by S.

Lemma 1. *Let y be an optimal solution to an LP relaxation of* MAX INDEPENDENT SET *that has the downward property, then $|V(G)| - y(G) \geq (1 - \epsilon)\nu(G)$*

Proof. Consider the standard LP for MAX INDEPENDENT SET (LP(6)) and denote its solution by z. It is well known that z can be transformed into a half-integral solution. Partition $V(G)$ to S_0, S_1, and $S_{1/2}$ according to the value of z on the vertices. Also, let $S_{\text{int}} = S_0 \cup S_1$, G_{int} be the induced subgraph on S_{int}, and $G_{1/2}$ the induced subgraph on $S_{1/2}$

We first argue that the restriction of z on S_{int} is the optimal fractional solution of LP(6) on S_{int}. To see that, let w be any fractional solution to LP(6) on S_{int} and let u be the extension of w to S according to z, that is u agrees with w on S_{int} and with z on $S_{1/2}$. We now show that $(z+u)/2$ is a solution to LP(6) on G: edges inside S_{int} as well as edges inside $S_{1/2}$ are satisfied by both z and u, and so also by $(z+u)/2$; edges between S_0 and $S_{1/2}$ sum to at most $1/2$ in z and at most $3/2$ in u, and so must sum to at most 1 on $(z+u)/2$. Since there are no edges between S_1 and $S_{1/2}$ in G we have that $(z+u)/2$ is a valid solution. Optimality of z implies that $z(S) \geq u(S)$ and hence $z(S_{\text{int}}) \geq w(S_{\text{int}})$. Of course the same holds for any vector which is a solution to a tightening of LP(6) on S_{int}. In particular

$$y(S_{\text{int}}) \leq z(S_{\text{int}}) = |S_1|. \tag{8}$$

The second fact we require is that maximum independent set in $G_{1/2}$ is smaller than the minimum vertex cover of this graph. Since the all-half vector is solution of LP(6) on $G_{1/2}$, it is also a solution of the standard vertex cover relaxation. But then

$$\nu(G_{1/2}) \geq z(S_{1/2}) \geq \alpha(G_{1/2}). \tag{9}$$

With inequalities (8) and (9) we can easily conclude

$$\begin{aligned}
n - y(G) &= |S_{\text{int}}| - y(S_{\text{int}}) + |S_{1/2}| - y(S_{1/2}) \\
&\geq |S_{\text{int}}| - |S_1| + |S_{1/2}| - (1+\epsilon)\alpha(G_{1/2}) \\
&= |S_0| + \nu(G_{1/2}) - \epsilon\alpha(G_{1/2}) \\
&\geq |S_0| + \nu(G_{1/2}) - \epsilon\nu(G_{1/2}) \\
&\geq \nu(G) - \epsilon\nu(G_{1/2}) \\
&\geq \nu(G) - \epsilon\nu(G)
\end{aligned}$$

where the second last inequality follows since the union of S_0 and any vertex cover of $G_{1/2}$ is a vertex cover for G.

For any graph G which is H minor-free all its subgraphs are also H minor-free. This fact shows that we satisfy the conditions of Lemma 1. Now if we use Theorem 3, we can immediately get that applying level c SA operator is sufficient to obtain the $n - y(G) \geq (1-\epsilon)\nu(G)$ inequality. Specifically, we have

Theorem 4. *After applying level k SA operator the above Linear program, we have a approximation of* $1 - 1/f(k)$ *for* MIN VERTEX COVER.

Any subgraph of a H minor-free graph is also a H minor-free graph and therefore it satisfies the second condition of Lemma 1. Also it is clear that it satisfies the first condition as SA is a tightening of the LP (6). and therefore the approximation on independent set follows the approximation of vertex cover for planar graphs.

5 Main result: a PTAS for Max Independent Set on Noisy Minor-Free Graphs

Algorithms that makes assumptions about the nature of their input may completely break down when this assumption is not totally met, even if by just a little. Indeed, try to two-colour a graph that is not quite two-colourable, or to approximate Max2SAT for formulas that are almost satisfiable by using an algorithm that solves 2SAT. Perhaps the most obvious example of this sort is MAX-2LIN, the problem of satisfying a maximal number of linear equations. This problem can be solved easily using Gaussian elimination if there is an assignment satisfying all equations but is hard to approximate when this is not the case, even when the system is nearly satisfiable.

Of course a better scenario is when the algorithms are *robust*. Such algorithms are designed to work well on a special class of inputs but even when the input slightly inconsistent with the class (of course, "slightly" should be well defined in some natural way) then the performance (approximation) of the algorithm may only deteriorate in some controlled way.

As was outlined in the Preliminaries, in the context of graphs we say that a graph is close to being Minor Free if by removing a small number of edges the obtained graph is minor-free. With this in mind, we would like to know whether there are good algorithms when the input graph is either minor-free or it can be made minor-free after, say, $o(n)$ edges are removed from.

We first argue that previous combinatorial algorithms, or even other algorithms that work in the same spirit, are non-robust. Notice that all previous algorithms relied on finding a decomposition of the graph into simpler (small tree-width) parts, in a manner which "resembles" a partition. For simplicity we will consider the spacial case of robustness with respect to planar graphs. Had there been robust combinatorial algorithms we would that along the way such algorithms will provide decomposition of the above nature. But then we should also expect such algorithms to perform the simpler task of deleting a few nodes and edges in such graphs so as to make them planar. Two relevant combinatorial problems come in mind, Maximum Planar Subgraph and Minimum Non-planar Deletion, the first asking to find a planar subgraph of the input graph G with maximum number of edges, and the second is the complementary problem, that is minimizing the number of edges to delete to make G planar. These problems are well studied and was shown to be APX-hard [9, 22].

In contrast, the Sherali-Adams based approach uses such decomposition *only in its analysis* and so the algorithmic difficulty in detecting the "wrong edges" disappears. Here is what we can obtain. We jump right away to the general minor-free case, although similar argument will provide an algorithm for the planar case with improved parameters.

Theorem 5. *For every H and ϵ, there exists a constant $r = r(\epsilon, H)$ such after applying level-r SA operator to LP (6) for* Max Independent Set *with input graph G which has distance $d = O(n/|H|\sqrt{log|H|})$ from an H-minor-free graph,* *the integrality gap is at most*

$$1 + \epsilon + O(d|H|\sqrt{\log |H|}/n).$$

Proof. Let F be an H-minor-free graph that is closest to G. It is easy to verify that (i) $V(F) \subseteq V(G)$ (ii) $E(F) \subseteq E(G)$ and further that $|E(G) - E(F)| \leq d$. Since the removal of every edge can increase the size of the maximum independent set by 1, and since the removal of an isolated vertex will decrease it by 1, it follows that

$$|\alpha(G) - \alpha(F)| \leq d.$$

The next structural statement we need in order to control the behaviour of G compared to that of F is the strength of $\mathrm{SA}^{(t)}(G)$ compared to that of $\mathrm{SA}^{(t)}(F)$. Let y be the optimal solution of $\mathrm{SA}^{(t)}(G)$. Since $E(F) \subseteq E(G)$ we can use the monotonicity argument as in the proof of Theorem refmain to deduce that the restriction of y to F is a valid solution to $\mathrm{SA}^{(t)}(F)$. This allows us to bound $y(F)$ as if it is obtained in $\mathrm{SA}^{(t)}(F)$ and hence we can use Theorem 3, which sys that there exists a constant $r = r(\epsilon, H)$ such that after applying level r Sherali-Adams operator, we get a bound

$$y(F) \leq (1 + \epsilon)\alpha(F). \tag{10}$$

Recall that $y(F)$ is just a projection of the vector y onto F, hence

$$y(G) - y(F) \leq d \tag{11}$$

We next argue that there are large independent sets in F. Indeed, recall that the greedy algorithm that repeatedly takes a vertex of lowest degree to the independent set and removes its neighbours, gives an independent set of size $\Omega(n/\delta)$ where δ is the average degree in F. It is known [21, 19, 1] that H-minor-free graphs have on average degree $O(|H|\sqrt{\log |H|})$, hence an independent set of size $\Omega(n/|H|\sqrt{\log |H|})$ is obtained. Since $d = O(n/|H|\sqrt{\log |H|})$ we get that $d = O(\alpha(F))$. We will assume from now on that the hidden constant is such that

$$d \leq \alpha(F)/4 \tag{12}$$

We now combine inequalities 10, 11 and 12 to get obtained the desired bound on the integrality gap of $\mathrm{SA}^{(t)}(G)$.

$$\begin{aligned}
\frac{y(G)}{\alpha(G)} &\leq \frac{y(F) + d}{\alpha(F) - d} \\
&\leq \frac{y(F)}{\alpha(F) - 2d} \\
&\leq \frac{(1 + \epsilon)\alpha(F)}{\alpha(F)(1 - 2d/\alpha(F))} \\
&\leq (1 + \epsilon)(1 + 4d/\alpha(F)) \\
&= 1 + \epsilon + O(d|H|\sqrt{\log |H|}/n).
\end{aligned}$$

When Min Vertex Cover is harder than Min Independent Set: Is it possible to import the above result to the Min Vertex Cover problem a-la Section 4? We give a strong evidence that the answer is negative. The idea is based on two simple facts. First, a graph on d vertices has distance d from the empty graph. Second, the addition of isolated vertices to a graph the optimal value of the vertex cover LP does not change, nd the same holds to the level r SA operator applied on that LP. By a result of Charikar, Makarychev and Makarychev [8] there are graphs on d nodes for which the integrality gap is $2 - o(1)$ even in the r-th level of the Sherali-Adams hierarchy for $r = d^{\Omega(1)}$. Specifically, the fractional solution (in the hierarchy) is roughly $d/2$ while the minimum vertex cover is $d(1 - o(1))$. Now, take a graph G_0 on d vertices as above and add $n - d$ isolated vertices to it. The obtained graph G will have (i) distance d from the empty graph on $n - d$ vertices (which is of course planar), and (ii) an optimal value of roughly $d/2$ in the $d^{\Omega(1)}$-level of the Sherali-Adams hierarchy. Thinking of d and n as asymptotically the same, say $d = n/100$ we get that even linear-level (in number of vertices) of Sherali-Adams has tight integrality-gap for graphs which are d distance away from planar graph, and so for the Min Vertex Cover problem, proximity to planarity does not preclude large integrality gaps.

6 Discussion

We have shown how LP-based algorithms "utilize" graph theoretical concepts in a different way compared to their combinatorial counterparts: While the combinatorial algorithms need to find a partition/decomposition of the graph in order to define the execution of the rest of the algorithm, in the Sherali-Adams world the special structure of the graph is used only in the analysis (at least for the problem of approximating the optimal value). This conceptual difference is what allows the Sherali-Admas approach to be successful where the combinatorial approach is limited.

In the introduction we have mentioned the Euclidean TSP result due to Arora[2]. Other works on connectivity problems for Planar/Euclidean case were since investigated, see [6, 7]. The underlying principle that is employed in these works is that a discretization of the space can approximate the problem well. The finer the discretization the better the approximation (at the cost of increased running time). Showing that a Sherali-Adams based algorithm leads to similar PTAS would be very interesting. Again, such a result will give rise to a very simple algorithm, placing "all difficulty" on the analysis.

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