Problem Set 3

CSC 463

Due by March 13, 2020, 2pm

1. A language \( L \) is in \( \text{coNP} \) if the complement \( \overline{L} \in \text{NP} \). It is generally believed that \( \text{NP} \neq \text{coNP} \), which implies that \( \text{P} \neq \text{NP} \).

   A Boolean formula \( \phi \) is a \text{tautology} if \( \phi \) evaluates to true on every possible truth assignment.

   Let \( \text{TAUT} = \{ \langle \phi \rangle | \phi \text{ is a tautology} \} \) be the set of Boolean tautologies. Show that \( \text{TAUT} \) is \( \text{coNP} \)-Complete (i.e. \( \text{TAUT} \in \text{coNP} \) and there is reduction \( A \leq_p \text{TAUT} \) for all \( A \in \text{coNP} \)).

   \textbf{Solution:} Checking that a Boolean formula \( \phi \) is not a tautology can be done in polynomial time by checking if \( \phi(x_1, \ldots, x_n) \) evaluates to false on a given truth assignment \( x_1, \ldots, x_n \).

   Hence \( \text{TAUT} \in \text{coNP} \).

   Note that \( \phi \) is an unsatisfiable formula if and only if its negation \( \overline{\phi} \) is a tautology. Hence, there is a polynomial time reduction \( \text{SAT} \leq_p \text{TAUT} \). Therefore, given \( A \in \text{coNP} \), we have a reduction \( A \leq_p \text{TAUT} \) by using the fact that there is a reduction \( \text{SAT} \leq_p \text{TAUT} \) using \( \text{NP} \)-Completeness of \( \text{SAT} \), and composing it with the reduction \( \text{SAT} \leq_p \text{TAUT} \). Therefore, \( \text{TAUT} \) is \( \text{coNP} \)-Complete.

2. A \text{partial 3-colouring} of a graph \( G \) is a map \( g : V' \mapsto \{1, 2, 3\} \) such that \( g(u) \neq g(v) \) for vertices \( u, v \in V' \) in some subset \( V' \subset V \) provided \((u, v)\) is an edge in \( G \). We say that \( g \) \text{extends} to a full 3-colouring of \( G \) if there is a 3-colouring \( f \) covering all vertices of \( G \) such that \( f(u) = g(u) \) for all \( u \in V' \). Let \( \text{PARTIAL-3COL} \) be the problem of determining given \( \langle G, g \rangle \) where \( g \) is a partial 3-colouring of \( G \), whether or not \( g \) extends to a full 3-colouring of \( G \).

   Show that there is a reduction \( \text{PARTIAL-3COL} \leq_p \text{3COL} \). Therefore, there is an efficient decision-to-search reduction for \( \text{3COL} \).

   \textbf{Solution:} Let \( \langle G, g \rangle \) be an instance of partial 3-colouring. Create a graph \( G' \) by adding 3 additional vertices labelled \( \{v_1, v_2, v_3\} \) connected as a triangle, and adding edges between \((v, v_j)\) with \( j \neq g(v) \) for every vertex where \( g(v) \) is defined. Clearly, \( G' \) can be created in polynomial time.

   Now we need to show that \( G' \) is 3-colourable if and only if \( g \) extends to a 3-colouring in \( G \). If \( g \) extends to a 3-colouring \( f \) of \( G \), the same colouring \( f \) with the new vertices \( v_i \) coloured with \( i \) is a proper colouring of \( G' \). Otherwise, suppose \( G' \) is 3-colourable with colouring \( f \). We are not guaranteed that \( f(v_1) = 1, f(v_2) = 2, f(v_3) = 3 \) since there in total 6 valid colourings of those vertices, but we can permute the colours of \( f \) to create a new colouring \( f' \) to make sure \( v_i \) is assigned colour \( i \). Note then that \( f' \) is then an extension of \( g \) that colours \( G \).

3. A Boolean formula \( \phi \) is \text{nice} if it is in conjunctive normal form, and each clause consists entirely of unnegated or negated variables. Let \( \text{NICE-SAT} \) be the problem of determining if a nice Boolean formula is satisfiable or not. Prove that \( \text{NICE-SAT} \) is \( \text{NP} \)-Complete.
Solution: Since satisfiability restricted to conjunctive normal form formulas is in \( \text{NP} \), we can conclude that \( \text{NICE-SAT} \in \text{NP} \) by using the same procedure to check if a given truth assignment satisfies a formula in polynomial time. Now we show a polynomial time reduction between \( \text{SAT} \) and \( \text{NICE-SAT} \) to complete the proof.

Given a CNF Boolean formula \( \phi \), we create a new Boolean formula \( \phi' \) that is nice by firstly replacing all negated variable \( x_i \) with a new unnegated variable \( y_i \). To ensure that \( y_i \) and \( x_i \) have the same truth assignments, we add the clauses \((y_i \lor x_i) \land (\neg y_i \lor \neg x_i)\) to \( \phi' \) which are satisfiable if and only if \( y_i = x_i \). Clearly the reduction produces a nice formula, and it can be done in polynomial time in the number of clauses of \( \phi \).

A satisfying assignment for \( x_1, \ldots, x_n \) in \( \phi \) can be extended to a satisfying assignment for \( \phi' \) by assigning \( y_i = \neg x_i \) for each new variable. Conversely, if \( \phi' \) is satisfiable, taking the truth values of \( x_1, \ldots, x_n \) in the satisfying assignment satisfies \( \phi \), since we have ensured that \( y_i = \neg x_i \) in any satisfying assignment.

4. Given a graph \( G \), we say that a subset of its vertices \( S \subset V \) is \textbf{triangle-free} if for every size three subset \( \{u, v, w\} \subset S \), at least one of the edges \( uv, vw, uw \) is not in the graph \( G \). Let \( \text{TF} \) be the problem of determining if a graph \( G \) has a triangle-free subset of size at least \( k \). Show that \( \text{TF} \in \text{NP} \)-Complete by reducing from independent set.

Solution: We can check if a subset \( S \subset V \) in \( G \) is triangle free in \( O(n^3) \) time where \( n \) is the number of vertices of \( G \). Hence \( \text{TF} \in \text{NP} \). Now we reduce independent-set to \( \text{TF} \) to prove hardness.

Given a graph \( G \) with \( n \) vertices \( V \), we create a new graph \( G' \) with vertices \( V \cup V' \) where \( V' \) has \( n + 1 \) new vertices. Furthermore, connect each new vertex \( v \in V' \) to each of the vertex in \( V \), and no pair of vertices in \( V' \) is connected. The reduction can be done in \( O(n^2) \) time. We claim that \( G \) has an independent set of size \( k \) if and only if \( G' \) has an triangle-free subset of size \( k + n + 1 \).
If $G$ has an independent set $I$ of size $k$, observe that $I \cup V'$ is triangle-free of size $k + n + 1$ since any triple of vertices must have two vertices in $I$ or two vertices in $V'$, but by construction $I$ and $V'$ by themselves are independent sets so at least one edge is missing in the triple.

Conversely, if $G'$ has a triangle-free subset $S$ of size $k + n + 1$, then it cannot contain two vertices $u, v \in V$ connected by an edge. This is because, if it did, then no vertex $w \in V'$ can be included since all edges are present between vertices in $V$ and $V'$ and this creates a triangle. Hence, $S$ contains only vertices in $V$, and we get that $k + n + 1 \leq |S| \leq n$, which is a contradiction to $k$ being positive. Hence, $S \cap V$ is an independent set of size at least $k$, after removing all possible vertices in $V'$ from $S$.

5. (Optional/ungraded: for extra knowledge and extra practice) Recall a weighted graph is a graph with a function $w : E \mapsto \mathbb{Z}^+$ assigning weights to its edges. Given a graph $G$, a cut is a division of the vertices into disjoint sets $(S, T)$ with $S \cup T = V$. An edge crosses the cut if one of its endpoints is in $S$ and the other endpoint is in $T$. The weight of a cut in a weighted graph is the sum $\sum_{e \in E} w(e)$ over all edges $e$ crossing the cut. Given a weighted graph, show that the problem of determining if it has a cut of weight at least $k$ is NP-Complete. This problem is related to the Ising model of magnetism studied in physics.

Solution: We use the fact that deciding an integer array has a partitioning is NP-Complete. Let $A$ be an array with positive integers $c_i \geq 0$ where $\sum_{i=1}^{n} c_i = k$. $A$ has a partitioning if you can find indices $I \subset [n] = \{1, \ldots, n\}$ where the elements at $I$ sum to $\sum_{i \in I} c_i = k/2$. Determining if $A$ has a partitioning is NP-Complete.

Given an array $A$ of $n$ elements with positive integer elements $c_i$ summing to $k$, we create a weighted graph $G_A$ that is complete with vertices labelled $V = \{1, \ldots, n\}$ where the edge between vertices $i, j$ is weighted by $c_i c_j$. Clearly this can be done in polynomial time in the length of the array. Now we need to show that $A$ has a partitioning if and only $G_A$ has a cut of weight at least $\frac{1}{4} k^2$.

Let $(I, [n] \setminus I)$ be any cut in $G_A$. Note that it has weight
\[
\sum_{i\in I} \sum_{j\in [n]\setminus I} c_i c_j = \left( \sum_{i\in I} c_i \right) \left( \sum_{j\in [n]\setminus I} c_j \right),
\]
by construction. Hence if a partitioning of \( A \) exists, there is a cut of weight \( \frac{k^2}{4} \). Otherwise, if \( G_A \) has a cut achieving weight at least \( \frac{k^2}{4} \), then there is a partitioning of \( A \) with \( \sum_{i\in I} c_i = \frac{k}{2} \) since assuming that \( a + b = k \), the maximum value of \( ab \) is \( \frac{k^2}{4} \) and is achieved when \( a = b = \frac{k}{2} \). Hence, we have found a partitioning of the elements in \( A \).

To relate this problem to the Ising model of magnetism, the Ising model says that \( n \) magnetic spins \( \sigma_i \) that can take \( \pm 1 \) values and interaction strengths \( J_{i,j} \) between the spins, a magnet will minimize the energy \( E(\sigma_1, \ldots, \sigma_n) = -\sum_{i,j=1}^{n} J_{i,j} \sigma_i \sigma_j \). To formulate this as a decision problem in \( \textbf{NP} \), we can ask if there is a configuration of the spins that achieves energy \( \leq k \) for some \( k \). A weighted graph can be created from the interaction strengths \( J_{i,j} \) such that the weight of the maximum cut corresponds to the minimum energy of the magnet.