A picture can be described in several ways. Assuming we have a raster display, a picture is completely specified by the set of intensities for the pixel positions in the display. At the other extreme, we can describe a picture as a set of complex objects, such as trees and terrain or furniture and walls, positioned at specified coordinate locations within the scene. Shapes and colors of the objects can be described internally with pixel arrays or with sets of basic geometric structures, such as straight line segments and polygon color areas. The scene is then displayed either by loading the pixel arrays into the frame buffer or by scan converting the basic geometric-structure specifications into pixel patterns. Typically, graphics programming packages provide functions to describe a scene in terms of these basic geometric structures, referred to as **output primitives**, and to group sets of output primitives into more complex structures. Each output primitive is specified with input coordinate data and other information about the way that object is to be displayed. Points and straight line segments are the simplest geometric components of pictures. Additional output primitives that can be used to construct a picture include circles and other conic sections, quadric surfaces, spline curves and surfaces, polygon color areas, and character strings. We begin our discussion of picture-generation procedures by examining device-level algorithms for displaying two-dimensional output primitives, with particular emphasis on scan-conversion methods for raster graphics systems. In this chapter, we also consider how output functions can be provided in graphics packages, and we take a look at the output functions available in the PHIGS language.

3-1 POINTS AND LINES

Point plotting is accomplished by converting a single coordinate position furnished by an application program into appropriate operations for the output device in use. With a CRT monitor, for example, the electron beam is turned on to illuminate the screen phosphor at the selected location. How the electron beam is positioned depends on the display technology. A random-scan (vector) system stores point-plotting instructions in the display list, and coordinate values in these instructions are converted to deflection voltages that position the electron beam at the screen locations to be plotted during each refresh cycle. For a black-and-white raster system, on the other hand, a point is plotted by setting the bit value corresponding to a specified screen position within the frame buffer to 1. Then, as the electron beam sweeps across each horizontal scan line, it emits a

burst of electrons (plots a point) whenever a value of 1 is encountered in the frame buffer. With an RGB system, the frame buffer is loaded with the color codes for the intensities that are to be displayed at the screen pixel positions.

Line drawing is accomplished by calculating intermediate positions along the line path between two specified endpoint positions. An output device is then directed to fill in these positions between the endpoints. For analog devices, such as a vector pen plotter or a random-scan display, a straight line can be drawn smoothly from one endpoint to the other. Linearly varying horizontal and vertical deflection voltages are generated that are proportional to the required changes in the x and y directions to produce the smooth line.

Digital devices display a straight line segment by plotting discrete points between the two endpoints. Discrete coordinate positions along the line path are calculated from the equation of the line. For a raster video display, the line color (intensity) is then loaded into the frame buffer at the corresponding pixel coordinates. Reading from the frame buffer, the video controller then "plots" the screen pixels. Screen locations are referenced with integer values, so plotted positions may only approximate actual line positions between two specified endpoints. A computed line position of (10.48, 20.51), for example, would be converted to pixel position (10, 21). This rounding of coordinate values to integers causes lines to be displayed with a stairstep appearance ("the jaggies"), as represented in Fig 3-1. The characteristic stairstep shape of raster lines is particularly noticeable on systems with low resolution, and we can improve their appearance somewhat by displaying them on high-resolution systems. More effective techniques for smoothing raster lines are based on adjusting pixel intensities along the line paths.

For the raster-graphics device-level algorithms discussed in this chapter, object positions are specified directly in integer device coordinates. For the time being, we will assume that pixel positions are referenced according to scan-line number and column number (pixel position across a scan line). This addressing scheme is illustrated in Fig. 3-2. Scan lines are numbered consecutively from 0, starting at the bottom of the screen; and pixel columns are numbered from 0, left to right across each scan line. In Section 3-10, we consider alternative pixel addressing schemes.

To load a specified color into the frame buffer at a position corresponding to column x along scan line y, we will assume we have available a low-level procedure of the form

```
setPixel (x, y)
```

Figure 3-1
Stairstep effect (jaggies) produced when a line is generated as a series of pixel positions.

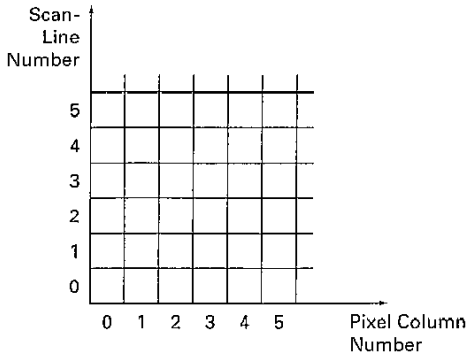


Figure 3-2
Pixel positions referenced by scan-line number and column number.

We sometimes will also want to be able to retrieve the current frame-buffer intensity setting for a specified location. We accomplish this with the low-level function

```
getPixel (x, y)
```

3-2 LINE-DRAWING ALGORITHMS

The Cartesian *slope-intercept equation* for a straight line is

$$y = m \cdot x + b \tag{3-1}$$

with m representing the slope of the line and b as the y intercept. Given that the two endpoints of a line segment are specified at positions (x_1, y_1) and (x_2, y_2), as shown in Fig. 3-3, we can determine values for the slope m and y intercept b with the following calculations:

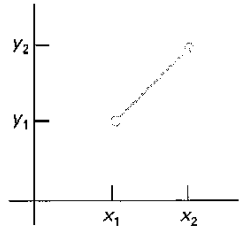


Figure 3-3
Line path between endpoint positions (x_1, y_1) and (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{3-2}$$

$$b = y_1 - m \cdot x_1 \tag{3-3}$$

Algorithms for displaying straight lines are based on the line equation 3-1 and the calculations given in Eqs. 3-2 and 3-3.

For any given x interval Δx along a line, we can compute the corresponding y interval Δy from Eq. 3-2 as

$$\Delta y = m \, \Delta x \tag{3-4}$$

Similarly, we can obtain the x interval Δx corresponding to a specified Δy as

$$\Delta x = \frac{\Delta y}{m} \tag{3-5}$$

These equations form the basis for determining deflection voltages in analog de-

vices. For lines with slope magnitudes $|m| < 1$, Δx can be set proportional to a small horizontal deflection voltage and the corresponding vertical deflection is then set proportional to Δy as calculated from Eq. 3-4. For lines whose slopes have magnitudes $|m| > 1$, Δy can be set proportional to a small vertical deflection voltage with the corresponding horizontal deflection voltage set proportional to Δx, calculated from Eq. 3-5. For lines with $m = 1$, $\Delta x = \Delta y$ and the horizontal and vertical deflections voltages are equal. In each case, a smooth line with slope m is generated between the specified endpoints.

On raster systems, lines are plotted with pixels, and step sizes in the horizontal and vertical directions are constrained by pixel separations. That is, we must "sample" a line at discrete positions and determine the nearest pixel to the line at each sampled position. This scan-conversion process for straight lines is illustrated in Fig. 3-4, for a near horizontal line with discrete sample positions along the x axis.

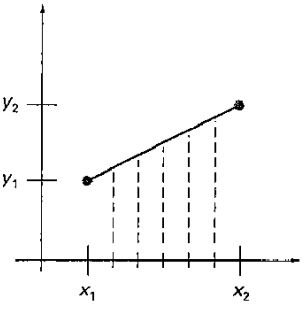


Figure 3-4
Straight line segment with five sampling positions along the x axis between x_1 and x_2.

DDA Algorithm

The *digital differential analyzer* (DDA) is a scan-conversion line algorithm based on calculating either Δy or Δx, using Eq. 3-4 or Eq. 3-5. We sample the line at unit intervals in one coordinate and determine corresponding integer values nearest the line path for the other coordinate.

Consider first a line with positive slope, as shown in Fig. 3-3. If the slope is less than or equal to 1, we sample at unit x intervals ($\Delta x = 1$) and compute each successive y value as

$$y_{k+1} = y_k + m \tag{3-6}$$

Subscript k takes integer values starting from 1, for the first point, and increases by 1 until the final endpoint is reached. Since m can be any real number between 0 and 1, the calculated y values must be rounded to the nearest integer.

For lines with a positive slope greater than 1, we reverse the roles of x and y. That is, we sample at unit y intervals ($\Delta y = 1$) and calculate each succeeding x value as

$$x_{k+1} = x_k + \frac{1}{m} \tag{3-7}$$

Equations 3-6 and 3-7 are based on the assumption that lines are to be processed from the left endpoint to the right endpoint (Fig. 3-3). If this processing is reversed, so that the starting endpoint is at the right, then either we have $\Delta x = -1$ and

$$y_{k+1} = y_k - m \tag{3-8}$$

or (when the slope is greater than 1) we have $\Delta y = -1$ with

$$x_{k+1} = x_k - \frac{1}{m} \tag{3-9}$$

Equations 3-6 through 3-9 can also be used to calculate pixel positions along a line with negative slope. If the absolute value of the slope is less than 1 and the start endpoint is at the left, we set $\Delta x = 1$ and calculate y values with Eq. 3-6.

When the start endpoint is at the right (for the same slope), we set $\Delta x = -1$ and obtain y positions from Eq. 3-8. Similarly, when the absolute value of a negative slope is greater than 1, we use $\Delta y = -1$ and Eq. 3-9 or we use $\Delta y = 1$ and Eq. 3-7.

This algorithm is summarized in the following procedure, which accepts as input the two endpoint pixel positions. Horizontal and vertical differences between the endpoint positions are assigned to parameters dx and dy. The difference with the greater magnitude determines the value of parameter `steps`. Starting with pixel position (x_a, y_a), we determine the offset needed at each step to generate the next pixel position along the line path. We loop through this process `steps` times. If the magnitude of dx is greater than the magnitude of dy and xa is less than xb, the values of the increments in the x and y directions are 1 and m, respectively. If the greater change is in the x direction, but xa is greater than xb, then the decrements -1 and $-m$ are used to generate each new point on the line. Otherwise, we use a unit increment (or decrement) in the y direction and an x increment (or decrement) of $1/m$.

```
#include "device.h"

#define ROUND(a) ((int)(a+0.5))

void lineDDA (int xa, int ya, int xb, int yb)
{
  int dx = xb - xa, dy = yb - ya, steps, k;
  float xIncrement, yIncrement, x = xa, y = ya;

  if (abs (dx) > abs (dy)) steps = abs (dx);
  else steps = abs (dy);
  xIncrement = dx / (float) steps;
  yIncrement = dy / (float) steps;

  setPixel (ROUND(x), ROUND(y));
  for (k=0; k<steps; k++) {
    x += xIncrement;
    y += yIncrement;
    setPixel (ROUND(x), ROUND(y));
  }
}
```

The DDA algorithm is a faster method for calculating pixel positions than the direct use of Eq. 3-1. It eliminates the multiplication in Eq. 3-1 by making use of raster characteristics, so that appropriate increments are applied in the x or y direction to step to pixel positions along the line path. The accumulation of roundoff error in successive additions of the floating-point increment, however, can cause the calculated pixel positions to drift away from the true line path for long line segments. Furthermore, the rounding operations and floating-point arithmetic in procedure `lineDDA` are still time-consuming. We can improve the performance of the DDA algorithm by separating the increments m and $1/m$ into integer and fractional parts so that all calculations are reduced to integer operations. A method for calculating $1/m$ increments in integer steps is discussed in Section 3-11. In the following sections, we consider more general scan-line procedures that can be applied to both lines and curves.

Bresenham's Line Algorithm

An accurate and efficient raster line-generating algorithm, developed by Bresen-

ham, scan converts lines using only incremental integer calculations that can be adapted to display circles and other curves. Figures 3-5 and 3-6 illustrate sections of a display screen where straight line segments are to be drawn. The vertical axes show scan-line positions, and the horizontal axes identify pixel columns. Sampling at unit x intervals in these examples, we need to decide which of two possible pixel positions is closer to the line path at each sample step. Starting from the left endpoint shown in Fig. 3-5, we need to determine at the next sample position whether to plot the pixel at position (11, 11) or the one at (11, 12). Similarly, Fig. 3-6 shows a negative slope line path starting from the left endpoint at pixel position (50, 50). In this one, do we select the next pixel position as (51, 50) or as (51, 49)? These questions are answered with Bresenham's line algorithm by testing the sign of an integer parameter, whose value is proportional to the difference between the separations of the two pixel positions from the actual line path.

To illustrate Bresenham's approach, we first consider the scan-conversion process for lines with positive slope less than 1. Pixel positions along a line path are then determined by sampling at unit x intervals. Starting from the left endpoint (x_0, y_0) of a given line, we step to each successive column (x position) and plot the pixel whose scan-line y value is closest to the line path. Figure 3-7 demonstrates the kth step in this process. Assuming we have determined that the pixel at (x_k, y_k) is to be displayed, we next need to decide which pixel to plot in column x_{k+1}. Our choices are the pixels at positions (x_k+1, y_k) and (x_k+1, y_k+1).

At sampling position x_k+1, we label vertical pixel separations from the mathematical line path as d_1 and d_2 (Fig. 3-8). The y coordinate on the mathematical line at pixel column position x_k+1 is calculated as

$$y = m(x_k + 1) + b \tag{3-10}$$

Then

$$\begin{aligned} d_1 &= y - y_k \\ &= m(x_k + 1) + b - y_k \end{aligned}$$

and

$$\begin{aligned} d_2 &= (y_k + 1) - y \\ &= y_k + 1 - m(x_k + 1) - b \end{aligned}$$

The difference between these two separations is

$$d_1 - d_2 = 2m(x_k + 1) - 2y_k + 2b - 1 \tag{3-11}$$

A decision parameter p_k for the kth step in the line algorithm can be obtained by rearranging Eq. 3-11 so that it involves only integer calculations. We accomplish this by substituting $m = \Delta y/\Delta x$, where Δy and Δx are the vertical and horizontal separations of the endpoint positions, and defining:

$$\begin{aligned} p_k &= \Delta x(d_1 - d_2) \\ &= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + c \end{aligned} \tag{3-12}$$

The sign of p_k is the same as the sign of $d_1 - d_2$, since $\Delta x > 0$ for our example. Parameter c is constant and has the value $2\Delta y + \Delta x(2b - 1)$, which is independent

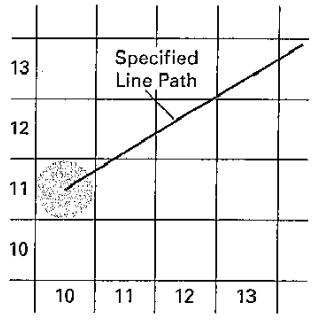


Figure 3-5
Section of a display screen where a straight line segment is to be plotted, starting from the pixel at column 10 on scan line 11.

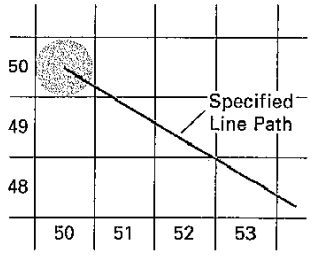
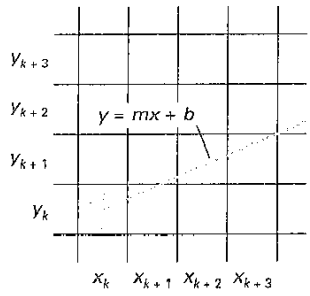
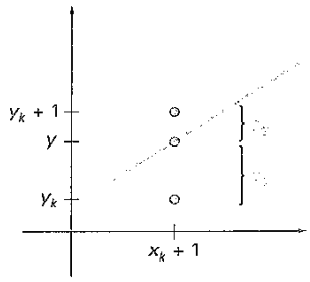


Figure 3-6
Section of a display screen where a negative slope line segment is to be plotted, starting from the pixel at column 50 on scan line 50.



Section of the screen grid showing a pixel in column x_k on scan line y_k that is to be plotted along the path of a line segment with slope $0 < m < 1$.



Distances between pixel positions and the line y coordinate at sampling position x_k+1.

of pixel position and will be eliminated in the recursive calculations for p_k. If the pixel at y_k is closer to the line path than the pixel at y_k+1 (that is, $d_1 < d_2$), then decision parameter p_k is negative. In that case, we plot the lower pixel; otherwise, we plot the upper pixel.

Coordinate changes along the line occur in unit steps in either the x or y directions. Therefore, we can obtain the values of successive decision parameters using incremental integer calculations. At step $k + 1$, the decision parameter is evaluated from Eq. 3-12 as

$$p_{k+1} = 2\Delta y \cdot x_{k+1} - 2\Delta x \cdot y_{k+1} + c$$

Subtracting Eq. 3-12 from the preceding equation, we have

$$p_{k+1} - p_k = 2\Delta y(x_{k+1} - x_k) - 2\,\Delta x(y_{k+1} - y_k)$$

But $x_{k+1} = x_k + 1$, so that

$$p_{k+1} = p_k + 2\Delta y - 2\Delta x(y_{k+1} - y_k)$$

where the term $y_{k+1} - y_k$ is either 0 or 1, depending on the sign of parameter p_k.

This recursive calculation of decision parameters is performed at each integer x position, starting at the left coordinate endpoint of the line. The first parameter, p_0, is evaluated from Eq. 3-12 at the starting pixel position (x_0, y_0) and with m evaluated as $\Delta y/\Delta x$:

$$p_0 = 2\Delta y - \Delta x$$

We can summarize Bresenham line drawing for a line with a positive slope less than 1 in the following listed steps. The constants $2\Delta y$ and $2\Delta y - 2\Delta x$ are calculated once for each line to be scan converted, so the arithmetic involves only integer addition and subtraction of these two constants.

Bresenham's Line-Drawing Algorithm for $|m| < 1$

1. Input the two line endpoints and store the left endpoint in (x_0, y_0).
2. Load (x_0, y_0) into the frame buffer; that is, plot the first point.
3. Calculate constants Δx, Δy, $2\Delta y$, and $2\Delta y - 2\Delta x$, and obtain the starting value for the decision parameter as

$$p_0 = 2\Delta y - \Delta x$$

4. At each x_k along the line, starting at $k = 0$, perform the following test: If $p_k < 0$, the next point to plot is $(x_k + 1, y_k)$ and

$$p_{k+1} = p_k + 2\Delta y$$

Otherwise, the next point to plot is $(x_k + 1, y_k + 1)$ and

$$p_{k+1} = p_k + 2\Delta y - 2\Delta x$$

5. Repeat step 4 Δx times.

Example 3-1 Bresenham Line Drawing

To illustrate the algorithm, we digitize the line with endpoints (20, 10) and (30, 18). This line has a slope of 0.8, with

$$\Delta x = 10, \qquad \Delta y = 8$$

The initial decision parameter has the value

$$\begin{aligned} p_0 &= 2\Delta y - \Delta x \\ &= 6 \end{aligned}$$

and the increments for calculating successive decision parameters are

$$2\Delta y = 16, \qquad 2\Delta y - 2\Delta x = -4$$

We plot the initial point $(x_0, y_0) = (20, 10)$, and determine successive pixel positions along the line path from the decision parameter as

k	p_k	(x_{k+1}, y_{k+1})	k	p_k	(x_{k+1}, y_{k+1})
0	6	(21, 11)	5	6	(26, 15)
1	2	(22, 12)	6	2	(27, 16)
2	−2	(23, 12)	7	−2	(28, 16)
3	14	(24, 13)	8	14	(29, 17)
4	10	(25, 14)	9	10	(30, 18)

A plot of the pixels generated along this line path is shown in Fig. 3-9.

An implementation of Bresenham line drawing for slopes in the range $0 < m < 1$ is given in the following procedure. Endpoint pixel positions for the line are passed to this procedure, and pixels are plotted from the left endpoint to the right endpoint. The call to `setPixel` loads a preset color value into the frame buffer at the specified (x, y) pixel position.

```
#include "device.h"

void lineBres (int xa, int ya, int xb, int yb)
{
  int dx = abs (xa - xb), dy = abs (ya - yb);
  int p = 2 * dy - dx;
  int twoDy = 2 * dy, twoDyDx = 2 * (dy - dx);
  int x, y, xEnd;

  /* Determine which point to use as start, which as end */
  if (xa > xb) {
    x = xb;
    y = yb;
    xEnd = xa;
  }
  else {
```

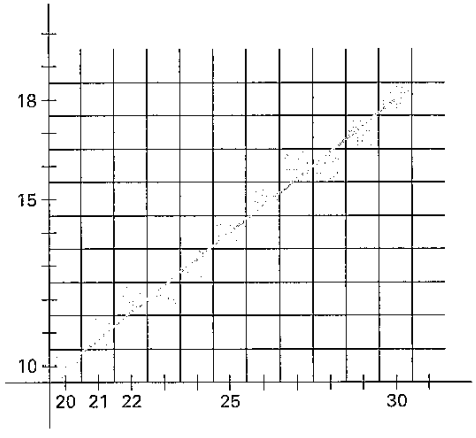
```
    x = xa;
    y = ya;
    xEnd = xb;
  }
  setPixel (x, y);

  while (x < xEnd) {
    x++;
    if (p < 0)
      p += twoDy;
    else {
      y++;
      p += twoDyDx;
    }
    setPixel (x, y);
  }
}
```

Bresenham's algorithm is generalized to lines with arbitrary slope by considering the symmetry between the various octants and quadrants of the xy plane. For a line with positive slope greater than 1, we interchange the roles of the x and y directions. That is, we step along the y direction in unit steps and calculate successive x values nearest the line path. Also, we could revise the program to plot pixels starting from either endpoint. If the initial position for a line with positive slope is the right endpoint, both x and y decrease as we step from right to left. To ensure that the same pixels are plotted regardless of the starting endpoint, we always choose the upper (or the lower) of the two candidate pixels whenever the two vertical separations from the line path are equal ($d_1 = d_2$). For negative slopes, the procedures are similar, except that now one coordinate decreases as the other increases. Finally, special cases can be handled separately: Horizontal lines ($\Delta y = 0$), vertical lines ($\Delta x = 0$), and diagonal lines with $|\Delta x| = |\Delta y|$ each can be loaded directly into the frame buffer without processing them through the line-plotting algorithm.

Parallel Line Algorithms

The line-generating algorithms we have discussed so far determine pixel positions sequentially. With a parallel computer, we can calculate pixel positions



Pixel positions along the line path between endpoints (20, 10) and (30, 18), plotted with Bresenham's line algorithm.

along a line path simultaneously by partitioning the computations among the various processors available. One approach to the partitioning problem is to adapt an existing sequential algorithm to take advantage of multiple processors. Alternatively, we can look for other ways to set up the processing so that pixel positions can be calculated efficiently in parallel. An important consideration in devising a parallel algorithm is to balance the processing load among the available processors.

Given n_p processors, we can set up a parallel Bresenham line algorithm by subdividing the line path into n_p partitions and simultaneously generating line segments in each of the subintervals. For a line with slope $0 < m < 1$ and left endpoint coordinate position (x_0, y_0), we partition the line along the positive x direction. The distance between beginning x positions of adjacent partitions can be calculated as

$$\Delta x_p = \frac{\Delta x + n_p - 1}{n_p} \tag{3-15}$$

where Δx is the width of the line, and the value for partition width Δx_p is computed using integer division. Numbering the partitions, and the processors, as 0, 1, 2, up to $n_p - 1$, we calculate the starting x coordinate for the kth partition as

$$x_k = x_0 + k\Delta x_p \tag{3-16}$$

As an example, suppose $\Delta x = 15$ and we have $n_p = 4$ processors. Then the width of the partitions is 4 and the starting x values for the partitions are x_0, $x_0 + 4$, $x_0 + 8$, and $x_0 + 12$. With this partitioning scheme, the width of the last (rightmost) subinterval will be smaller than the others in some cases. In addition, if the line endpoints are not integers, truncation errors can result in variable width partitions along the length of the line.

To apply Bresenham's algorithm over the partitions, we need the initial value for the y coordinate and the initial value for the decision parameter in each partition. The change Δy_p in the y direction over each partition is calculated from the line slope m and partition width Δx_p:

$$\Delta y_p = m\Delta x_p \tag{3-17}$$

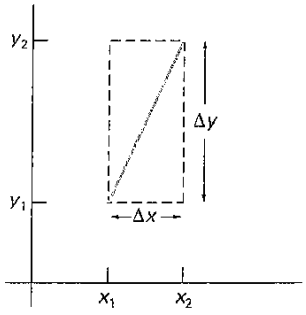
At the kth partition, the starting y coordinate is then

$$y_k = y_0 + \text{round}(k\Delta y_p) \tag{3-18}$$

The initial decision parameter for Bresenham's algorithm at the start of the kth subinterval is obtained from Eq. 3-12:

$$p_k = (k\Delta x_p)(2\Delta y) - \text{round}(k\Delta y_p)(2\Delta x) + 2\Delta y - \Delta x \tag{3-19}$$

Each processor then calculates pixel positions over its assigned subinterval using the starting decision parameter value for that subinterval and the starting coordinates (x_k, y_k). We can also reduce the floating-point calculations to integer arithmetic in the computations for starting values y_k and p_k by substituting $m = \Delta y/\Delta x$ and rearranging terms. The extension of the parallel Bresenham algorithm to a line with slope greater than 1 is achieved by partitioning the line in the y di-



Bounding box for a line with coordinate extents Δx and Δy.

rection and calculating beginning x values for the partitions. For negative slopes, we increment coordinate values in one direction and decrement in the other.

Another way to set up parallel algorithms on raster systems is to assign each processor to a particular group of screen pixels. With a sufficient number of processors (such as a Connection Machine CM-2 with over 65,000 processors), we can assign each processor to one pixel within some screen region. This approach can be adapted to line display by assigning one processor to each of the pixels within the limits of the line coordinate extents (*bounding rectangle*) and calculating pixel distances from the line path. The number of pixels within the bounding box of a line is $\Delta x \cdot \Delta y$ (Fig. 3-10). Perpendicular distance d from the line in Fig. 3-10 to a pixel with coordinates (x, y) is obtained with the calculation

$$d = Ax + By + C$$

where

$$A = \frac{-\Delta y}{\text{linelength}}$$

$$B = \frac{\Delta x}{\text{linelength}}$$

$$C = \frac{x_0 \Delta y - y_0 \Delta x}{\text{linelength}}$$

with

$$\text{linelength} = \sqrt{\Delta x^2 + \Delta y^2}$$

Once the constants A, B, and C have been evaluated for the line, each processor needs to perform two multiplications and two additions to compute the pixel distance d. A pixel is plotted if d is less than a specified line-thickness parameter.

Instead of partitioning the screen into single pixels, we can assign to each processor either a scan line or a column of pixels depending on the line slope. Each processor then calculates the intersection of the line with the horizontal row or vertical column of pixels assigned that processor. For a line with slope $|m| < 1$, each processor simply solves the line equation for y, given an x column value. For a line with slope magnitude greater than 1, the line equation is solved for x by each processor, given a scan-line y value. Such direct methods, although slow on sequential machines, can be performed very efficiently using multiple processors.

3-3 LOADING THE FRAME BUFFER

When straight line segments and other objects are scan converted for display with a raster system, frame-buffer positions must be calculated. We have assumed that this is accomplished with the `setPixel` procedure, which stores intensity values for the pixels at corresponding addresses within the frame-buffer array. Scan-conversion algorithms generate pixel positions at successive unit in-

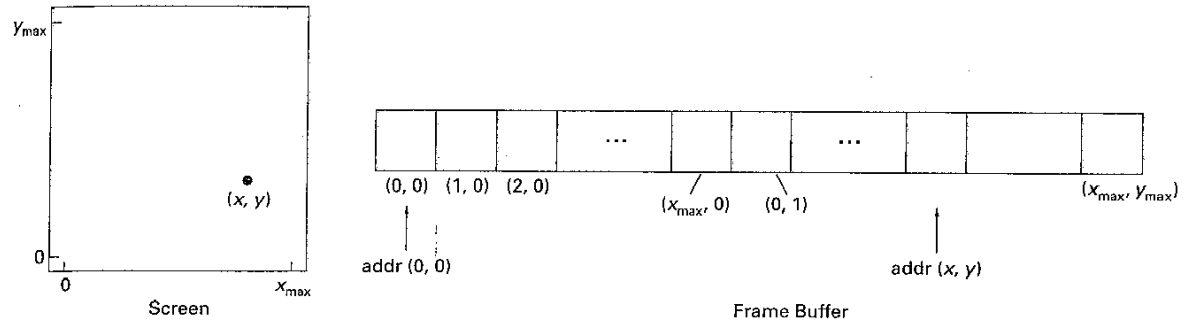


Figure 3-11
Pixel screen positions stored linearly in row-major order within the frame buffer.

tervals. This allows us to use incremental methods to calculate frame-buffer addresses.

As a specific example, suppose the frame-buffer array is addressed in row-major order and that pixel positions vary from (0, 0) at the lower left screen corner to (x_{max}, y_{max}) at the top right corner (Fig. 3-11). For a bilevel system (1 bit per pixel), the frame-buffer bit address for pixel position (x, y) is calculated as

$$\text{addr}(x, y) = \text{addr}(0, 0) + y(x_{max} + 1) + x \tag{3-21}$$

Moving across a scan line, we can calculate the frame-buffer address for the pixel at $(x + 1, y)$ as the following offset from the address for position (x, y):

$$\text{addr}(x + 1, y) = \text{addr}(x, y) + 1 \tag{3-22}$$

Stepping diagonally up to the next scan line from (x, y), we get to the frame-buffer address of $(x + 1, y + 1)$ with the calculation

$$\text{addr}(x + 1, y + 1) = \text{addr}(x, y) + x_{max} + 2 \tag{3-23}$$

where the constant $x_{max} + 2$ is precomputed once for all line segments. Similar incremental calculations can be obtained from Eq. 3-21 for unit steps in the negative x and y screen directions. Each of these address calculations involves only a single integer addition.

Methods for implementing the `setPixel` procedure to store pixel intensity values depend on the capabilities of a particular system and the design requirements of the software package. With systems that can display a range of intensity values for each pixel, frame-buffer address calculations would include pixel width (number of bits), as well as the pixel screen location.

3-4 LINE FUNCTION

A procedure for specifying straight-line segments can be set up in a number of different forms. In PHIGS, GKS, and some other packages, the two-dimensional line function is

```
polyline (n, wcPoints)
```

where parameter n is assigned an integer value equal to the number of coordinate positions to be input, and wcPoints is the array of input world-coordinate values for line segment endpoints. This function is used to define a set of $n - 1$ connected straight line segments. Because series of connected line segments occur more often than isolated line segments in graphics applications, polyline provides a more general line function. To display a single straight-line segment, we set $n = 2$ and list the x and y values of the two endpoint coordinates in wcPoints.

As an example of the use of polyline, the following statements generate two connected line segments, with endpoints at (50, 100), (150, 250), and (250, 100):

```
wcPoints[1].x = 50;
wcPoints[1].y = 100;
wcPoints[2].x = 150;
wcPoints[2].y = 250;
wcPoints[3].x = 250;
wcPoints[3].y = 100;
polyline (3, wcPoints);
```

Coordinate references in the polyline function are stated as **absolute coordinate** values. This means that the values specified are the actual point positions in the coordinate system in use.

Some graphics systems employ line (and point) functions with **relative coordinate** specifications. In this case, coordinate values are stated as offsets from the last position referenced (called the **current position**). For example, if location (3, 2) is the last position that has been referenced in an application program, a relative coordinate specification of (2, −1) corresponds to an absolute position of (5, 1). An additional function is also available for setting the current position before the line routine is summoned. With these packages, a user lists only the single pair of offsets in the line command. This signals the system to display a line starting from the current position to a final position determined by the offsets. The current position is then updated to this final line position. A series of connected lines is produced with such packages by a sequence of line commands, one for each line section to be drawn. Some graphics packages provide options allowing the user to specify line endpoints using either relative or absolute coordinates.

Implementation of the polyline procedure is accomplished by first performing a series of coordinate transformations, then making a sequence of calls to a device-level line-drawing routine. In PHIGS, the input line endpoints are actually specified in modeling coordinates, which are then converted to world coordinates. Next, world coordinates are converted to normalized coordinates, then to device coordinates. We discuss the details for carrying out these two-dimensional coordinate transformations in Chapter 6. Once in device coordinates, we display the polyline by invoking a line routine, such as Bresenham's algorithm, $n - 1$ times to connect the n coordinate points. Each successive call passes the coordinate pair needed to plot the next line section, where the first endpoint of each coordinate pair is the last endpoint of the previous section. To avoid setting the intensity of some endpoints twice, we could modify the line algorithm so that the last endpoint of each segment is not plotted. We discuss methods for avoiding overlap of displayed objects in more detail in Section 3-10.

3-5 CIRCLE-GENERATING ALGORITHMS

Since the circle is a frequently used component in pictures and graphs, a procedure for generating either full circles or circular arcs is included in most graphics packages. More generally, a single procedure can be provided to display either circular or elliptical curves.

Properties of Circles

A circle is defined as the set of points that are all at a given distance r from a center position (x_c, y_c) (Fig. 3-12). This distance relationship is expressed by the Pythagorean theorem in Cartesian coordinates as

$$(x - x_c)^2 + (y - y_c)^2 = r^2 \tag{3-24}$$

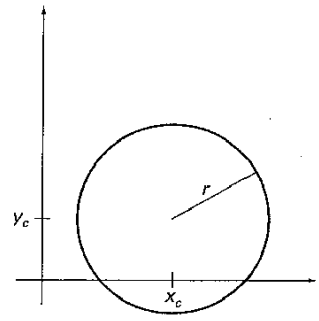


Figure 3-12
Circle with center coordinates (x_c, y_c) and radius r.

We could use this equation to calculate the position of points on a circle circumference by stepping along the x axis in unit steps from $x_c - r$ to $x_c + r$ and calculating the corresponding y values at each position as

$$y = y_c \pm \sqrt{r^2 - (x_c - x)^2} \tag{3-25}$$

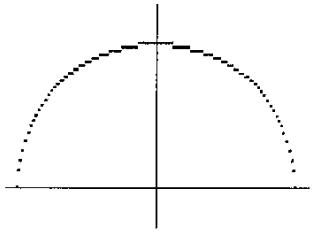


Figure 3-13
Positive half of a circle plotted with Eq. 3-25 and with $(x_c, y_c) = (0, 0)$.

But this is not the best method for generating a circle. One problem with this approach is that it involves considerable computation at each step. Moreover, the spacing between plotted pixel positions is not uniform, as demonstrated in Fig. 3-13. We could adjust the spacing by interchanging x and y (stepping through y values and calculating x values) whenever the absolute value of the slope of the circle is greater than 1. But this simply increases the computation and processing required by the algorithm.

Another way to eliminate the unequal spacing shown in Fig. 3-13 is to calculate points along the circular boundary using polar coordinates r and θ (Fig. 3-12). Expressing the circle equation in parametric polar form yields the pair of equations

$$\begin{aligned} x &= x_c + r\cos\theta \\ y &= y_c + r\sin\theta \end{aligned} \tag{3-26}$$

When a display is generated with these equations using a fixed angular step size, a circle is plotted with equally spaced points along the circumference. The step size chosen for θ depends on the application and the display device. Larger angular separations along the circumference can be connected with straight line segments to approximate the circular path. For a more continuous boundary on a raster display, we can set the step size at $1/r$. This plots pixel positions that are approximately one unit apart.

Computation can be reduced by considering the symmetry of circles. The shape of the circle is similar in each quadrant. We can generate the circle section in the second quadrant of the xy plane by noting that the two circle sections are symmetric with respect to the y axis. And circle sections in the third and fourth quadrants can be obtained from sections in the first and second quadrants by

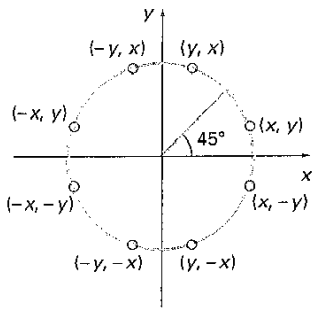


Figure 3-24
Symmetry of a circle. Calculation of a circle point (x, y) in one octant yields the circle points shown for the other seven octants.

considering symmetry about the x axis. We can take this one step further and note that there is also symmetry between octants. Circle sections in adjacent octants within one quadrant are symmetric with respect to the 45° line dividing the two octants. These symmetry conditions are illustrated in Fig.3-14, where a point at position (x, y) on a one-eighth circle sector is mapped into the seven circle points in the other octants of the xy plane. Taking advantage of the circle symmetry in this way, we can generate all pixel positions around a circle by calculating only the points within the sector from $x = 0$ to $x = y$.

Determining pixel positions along a circle circumference using either Eq. 3-24 or Eq. 3-26 still requires a good deal of computation time. The Cartesian equation 3-24 involves multiplications and square-root calculations, while the parametric equations contain multiplications and trigonometric calculations. More efficient circle algorithms are based on incremental calculation of decision parameters, as in the Bresenham line algorithm, which involves only simple integer operations.

Bresenham's line algorithm for raster displays is adapted to circle generation by setting up decision parameters for finding the closest pixel to the circumference at each sampling step. The circle equation 3-24, however, is nonlinear, so that square-root evaluations would be required to compute pixel distances from a circular path. Bresenham's circle algorithm avoids these square-root calculations by comparing the squares of the pixel separation distances.

A method for direct distance comparison is to test the halfway position between two pixels to determine if this midpoint is inside or outside the circle boundary. This method is more easily applied to other conics; and for an integer circle radius, the midpoint approach generates the same pixel positions as the Bresenham circle algorithm. Also, the error involved in locating pixel positions along any conic section using the midpoint test is limited to one-half the pixel separation.

Midpoint Circle Algorithm

As in the raster line algorithm, we sample at unit intervals and determine the closest pixel position to the specified circle path at each step. For a given radius r and screen center position (x_c, y_c), we can first set up our algorithm to calculate pixel positions around a circle path centered at the coordinate origin (0, 0). Then each calculated position (x, y) is moved to its proper screen position by adding x_c to x and y_c to y. Along the circle section from $x = 0$ to $x = y$ in the first quadrant, the slope of the curve varies from 0 to -1. Therefore, we can take unit steps in the positive x direction over this octant and use a decision parameter to determine which of the two possible y positions is closer to the circle path at each step. Positions in the other seven octants are then obtained by symmetry.

To apply the midpoint method, we define a circle function:

$$f_{\text{circle}}(x, y) = x^2 + y^2 - r^2 \qquad (3-\ldots)$$

Any point (x, y) on the boundary of the circle with radius r satisfies the equation $f_{\text{circle}}(x, y) = 0$. If the point is in the interior of the circle, the circle function is negative. And if the point is outside the circle, the circle function is positive. To summarize, the relative position of any point (x, y) can be determined by checking the sign of the circle function:

$$f_{\text{circle}}(x, y) \begin{cases} < 0, & \text{if } (x, y) \text{ is inside the circle boundary} \\ = 0, & \text{if } (x, y) \text{ is on the circle boundary} \\ > 0, & \text{if } (x, y) \text{ is outside the circle boundary} \end{cases} \tag{3-28}$$

The circle-function tests in 3-28 are performed for the midpositions between pixels near the circle path at each sampling step. Thus, the circle function is the decision parameter in the midpoint algorithm, and we can set up incremental calculations for this function as we did in the line algorithm.

Figure 3-15 shows the midpoint between the two candidate pixels at sampling position $x_k + 1$. Assuming we have just plotted the pixel at (x_k, y_k), we next need to determine whether the pixel at position $(x_k + 1, y_k)$ or the one at position $(x_k + 1, y_k - 1)$ is closer to the circle. Our decision parameter is the circle function 3-27 evaluated at the midpoint between these two pixels:

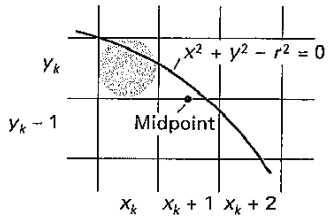


Figure 3-15
Midpoint between candidate pixels at sampling position x_k+1 along a circular path.

$$\begin{aligned} p_k &= f_{\text{circle}}\left(x_k + 1, y_k - \frac{1}{2}\right) \\ &= (x_k + 1)^2 + \left(y_k - \frac{1}{2}\right)^2 - r^2 \end{aligned} \tag{3-29}$$

If $p_k < 0$, this midpoint is inside the circle and the pixel on scan line y_k is closer to the circle boundary. Otherwise, the midposition is outside or on the circle boundary, and we select the pixel on scanline $y_k - 1$.

Successive decision parameters are obtained using incremental calculations. We obtain a recursive expression for the next decision parameter by evaluating the circle function at sampling position $x_{k+1} + 1 = x_k + 2$:

$$\begin{aligned} p_{k+1} &= f_{\text{circle}}\left(x_{k+1} + 1, y_{k+1} - \frac{1}{2}\right) \\ &= [(x_k + 1) + 1]^2 + \left(y_{k+1} - \frac{1}{2}\right)^2 - r^2 \end{aligned}$$

or

$$p_{k+1} = p_k + 2(x_k + 1) + (y_{k+1}^2 - y_k^2) - (y_{k+1} - y_k) + 1 \tag{3-30}$$

where y_{k+1} is either y_k or y_{k-1}, depending on the sign of p_k.

Increments for obtaining p_{k+1} are either $2x_{k+1} + 1$ (if p_k is negative) or $2x_{k+1} + 1 - 2y_{k+1}$. Evaluation of the terms $2x_{k+1}$ and $2y_{k+1}$ can also be done incrementally as

$$\begin{aligned} 2x_{k+1} &= 2x_k + 2 \\ 2y_{k+1} &= 2y_k - 2 \end{aligned}$$

At the start position $(0, r)$, these two terms have the values 0 and $2r$, respectively. Each successive value is obtained by adding 2 to the previous value of $2x$ and subtracting 2 from the previous value of $2y$.

The initial decision parameter is obtained by evaluating the circle function at the start position $(x_0, y_0) = (0, r)$:

$$p_0 = f_{\text{circle}}\left(1, r - \frac{1}{2}\right)$$
$$= 1 + \left(r - \frac{1}{2}\right)^2 - r^2$$

or

$$p_0 = \frac{5}{4} - r$$

If the radius r is specified as an integer, we can simply round p_0 to

$$p_0 = 1 - r \qquad \text{(for } r \text{ an integer)}$$

since all increments are integers.

As in Bresenham's line algorithm, the midpoint method calculates pixel positions along the circumference of a circle using integer additions and subtractions, assuming that the circle parameters are specified in integer screen coordinates. We can summarize the steps in the midpoint circle algorithm as follows.

Midpoint Circle Algorithm

1. Input radius r and circle center (x_c, y_c), and obtain the first point on the circumference of a circle centered on the origin as

$$(x_0, y_0) = (0, r)$$

2. Calculate the initial value of the decision parameter as

$$p_0 = \frac{5}{4} - r$$

3. At each x_k position, starting at $k = 0$, perform the following test: If $p_k < 0$, the next point along the circle centered on (0, 0) is (x_{k+1}, y_k) and

$$p_{k+1} = p_k + 2x_{k+1} + 1$$

   Otherwise, the next point along the circle is $(x_k + 1, y_k - 1)$ and

$$p_{k+1} = p_k + 2x_{k+1} + 1 - 2y_{k+1}$$

   where $2x_{k+1} = 2x_k + 2$ and $2y_{k+1} = 2y_k - 2$.

4. Determine symmetry points in the other seven octants.
5. Move each calculated pixel position (x, y) onto the circular path centered on (x_c, y_c) and plot the coordinate values:

$$x = x + x_c, \qquad y = y + y_c$$

6. Repeat steps 3 through 5 until $x \geq y$.

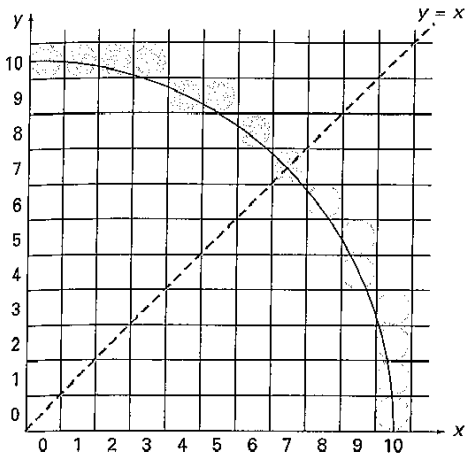


Figure 3-16
Selected pixel positions (solid circles) along a circle path with radius $r = 10$ centered on the origin, using the midpoint circle algorithm. Open circles show the symmetry positions in the first quadrant.

Example 3-2 Midpoint Circle-Drawing

Given a circle radius $r = 10$, we demonstrate the midpoint circle algorithm by determining positions along the circle octant in the first quadrant from $x = 0$ to $x = y$. The initial value of the decision parameter is

$$p_0 = 1 - r = -9$$

For the circle centered on the coordinate origin, the initial point is $(x_0, y_0) = (0, 10)$, and initial increment terms for calculating the decision parameters are

$$2x_0 = 0, \qquad 2y_0 = 20$$

Successive decision parameter values and positions along the circle path are calculated using the midpoint method as

k	p_k	(x_{k+1}, y_{k+1})	$2x_{k+1}$	$2y_{k+1}$
0	−9	(1, 10)	2	20
1	−6	(2, 10)	4	20
2	−1	(3, 10)	6	20
3	6	(4, 9)	8	18
4	−3	(5, 9)	10	18
5	8	(6, 8)	12	16
6	5	(7, 7)	14	14

A plot of the generated pixel positions in the first quadrant is shown in Fig. 3-16.

The following procedure displays a raster circle on a bilevel monitor using the midpoint algorithm. Input to the procedure are the coordinates for the circle center and the radius. Intensities for pixel positions along the circle circumference are loaded into the frame-buffer array with calls to the `setPixel` routine.

```
#include "device.h"

void circleMidpoint (int xCenter, int yCenter, int radius)
{
  int x = 0;
  int y = radius;
  int p = 1 - radius;
  void circlePlotPoints (int, int, int, int);

  /* Plot first set of points */
  circlePlotPoints (xCenter, yCenter, x, y);

  while (x < y) {
    x++;
    if (p < 0)
      p += 2 * x + 1;
    else {
      y--;
      p += 2 * (x - y) + 1;
    }
    circlePlotPoints (xCenter, yCenter, x, y);
  }
}

void circlePlotPoints (int xCenter, int yCenter, int x, int y)
{
  setPixel (xCenter + x, yCenter + y);
  setPixel (xCenter - x, yCenter + y);
  setPixel (xCenter + x, yCenter - y);
  setPixel (xCenter - x, yCenter - y);
  setPixel (xCenter + y, yCenter + x);
  setPixel (xCenter - y, yCenter + x);
  setPixel (xCenter + y, yCenter - x);
  setPixel (xCenter - y, yCenter - x);
}
```

3-6 ELLIPSE-GENERATING ALGORITHMS

Loosely stated, an ellipse is an elongated circle. Therefore, elliptical curves can be generated by modifying circle-drawing procedures to take into account the different dimensions of an ellipse along the major and minor axes.

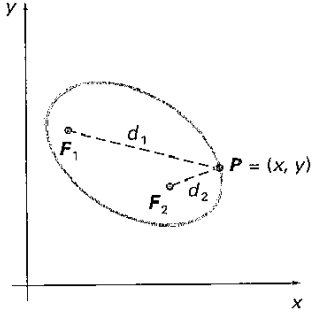


Figure 3-17
Ellipse generated about foci F_1 and F_2.

Properties of Ellipses

An ellipse is defined as the set of points such that the sum of the distances from two fixed positions (foci) is the same for all points (Fig. 3-17). If the distances to the two foci from any point $\mathbf{P} = (x, y)$ on the ellipse are labeled d_1 and d_2, then the general equation of an ellipse can be stated as

$$d_1 + d_2 = \text{constant} \qquad (3\text{-}2)$$

Expressing distances d_1 and d_2 in terms of the focal coordinates $\mathbf{F}_1 = (x_1, y_1)$ and $\mathbf{F}_2 = (x_2, y_2)$, we have

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = \text{constant} \qquad (3\text{-}3)$$