Rounding and Approximation Algorithms

CSC 473 Advanced Algorithms



Approximation Algorithms

- Many natural and important optimization problems are NP-Hard
 - Vertex/Set cover, Max SAT, Max Cut, Sparsest Cut, Traveling Salesman, ...
- No worst-case polynomial time exact algorithms
- Ways around the hardness:
 - Approximation algorithms: output an approximately optimal solution in worst-case polynomial time
 - Algorithms that are efficient on special instances: e.g. Max Cut in planar graphs
 - Algorithms that are exponential in some parameter: $2^{O(k)}poly(n)$ time algorithm to find a path of length k.



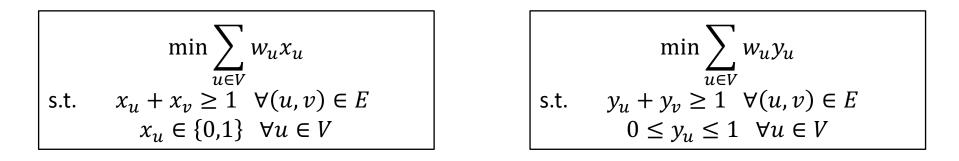
Vertex Cover

- Min Vertex Cover (VC): on input a graph G = (V, E) and vertex weights $w \in \mathbb{R}^V$ find set $C \subseteq V$ such that
 - for all $(u, v) \in E$, $u \in C$, or $v \in C$, or both
 - $w(C) = \sum_{u \in C} w_v$ is minimized
- NP-hard for non-bipartite graphs
- Factor 2 approximation
 - Formulate as an IP, and relax to an LP
 - Round (possibly fractional) optimal LP solution to a $\{0,1\}$ solution



LP Relaxation

• Let $OPT(G, w) = \min\{w(C): C \text{ is a vertex cover}\}$



- LP(G, w) = value of the LP
- OPT(G, w) =value of the IP $\geq LP(G, w)$



Deterministic Rounding

- Will show: $LP(G, w) \leq OPT(G, w) \leq 2 \cdot LP(G, w)$
- Let y be an optimal LP solution and define

$$C = \{u \in V \colon y_u \ge 1/2\}$$

$$\min \sum_{u \in V} w_u y_u$$

s.t. $y_u + y_v \ge 1 \quad \forall (u, v) \in E$
 $0 \le y_u \le 1 \quad \forall u \in V$

- *C* is a vertex cover
 - If $(u, v) \in E$ then $\max\{y_u, y_v\} \ge \frac{1}{2}$
- *C* is a 2-approximate solution: Let $x_u = 1 \Leftrightarrow u \in C$. Then $x_u \le 2y_u$ and

$$w(C) = \sum_{u \in V} w_u x_u \le 2 \sum_{u \in V} w_u y_u = 2 \cdot LP(G, w) \le 2 \cdot OPT(G, w)$$



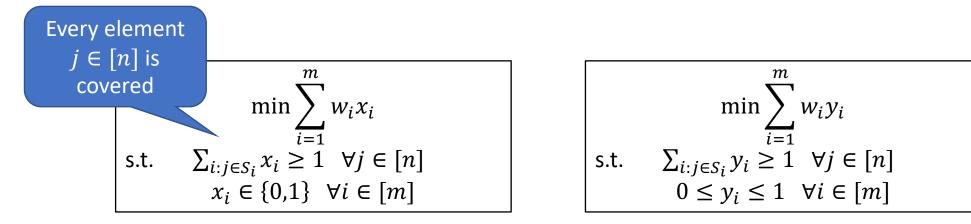
Set Cover

- Min Set Cover: on input $S_1, S_2, ..., S_m \subseteq [n]$ where $\bigcup_{i=1}^m S_i = [n]$, and weights $w \in \mathbb{R}^m$, find set $C \subseteq [m]$ such that
 - $\bigcup_{i \in C} S_i = [n]$
 - $w(C) = \sum_{i \in C} w_i$ is minimized
- Vertex cover is a special case
- Factor $O(\log n)$ approximation
 - No better factor possible, unless P=NP



LP Relaxation

• Let $OPT = \min\{w(C): C \text{ is a vertex cover}\}$



- LP = value of the LP
- OPT = value of the IP $\geq LP$



Randomized Rounding

s.t.
$$\sum_{i:j\in S_i}^m y_i \ge 1 \quad \forall j \in [n]$$
$$0 \le y_i \le 1 \quad \forall i \in [m]$$

- y = optimal LP solution
- For $t = 1, \dots, \ell = \ln(2n)$
 - $C_t = \emptyset$
 - For i = 1, ..., m: Add i to C_t with probability y_i
- $C = C_1 \cup \cdots \cup C_\ell$
- Claim 1. $\mathbb{E}[w(C)] \leq \ell \cdot LP$
 - $Z_{t,i} = 1$ if $i \in C_t$, 0 otherwise. $\mathbb{E}[Z_{t,i}] = \mathbb{P}(Z_{t,i} = 1) = y_i$.

$$\mathbb{E}[w(C)] \leq \mathbb{E}[w(C_1)] + \dots + \mathbb{E}[w(C_\ell)] = \sum_{t=1}^\ell \sum_{i=1}^m w_i \mathbb{E}[Z_{t,i}] = \ell \sum_{i=1}^m w_i y_i$$



Randomized Rounding

s.t.
$$\sum_{i:j\in S_i}^m y_i \ge 1 \quad \forall j \in [n]$$
$$0 \le y_i \le 1 \quad \forall i \in [m]$$

- y = optimal LP solution
- For $t = 1, \dots, \ell = \ln(2n)$

• $C_t = \emptyset$

- For i = 1, ..., m: Add i to C_t with probability y_i
- $C = C_1 \cup \cdots \cup C_\ell$
- Claim 2. $\mathbb{P}(C \text{ is a set cover}) \geq \frac{1}{2}$ $\forall j: \mathbb{P}\left(\forall t: j \notin \bigcup_{i \in C_t} S_i\right) = \prod_{t=1}^{\ell} \prod_{i:j \in S_i} (1-y_i) \leq e^{-\ell \sum_{i:j \in S_i} y_i} \leq e^{-\ell} = \frac{1}{2n}$

• By a union bound, the probability that some j is not covered is $\leq n \cdot \frac{1}{2n} = \frac{1}{2}$ Computer Science

Randomized Rounding

What if w(C) is small only when C is not a cover?

- Claim 1. $\mathbb{E}[w(C)] \leq \ell \cdot LP$
- Claim 2. $\mathbb{P}(C \text{ is a set cover}) \geq \frac{1}{2}$
- Repeat until C is a set cover. Expected weight is $\mathbb{E}[w(C)|C$ a cover]

 $\mathbb{E}[w(C)] = \mathbb{E}[w(C)|C \text{ a cover}]\mathbb{P}(C \text{ a cover})$

 $+\mathbb{E}[w(C)|C \text{ not a cover}]\mathbb{P}(C \text{ not a cover}) \\ \geq \mathbb{E}[w(C)|C \text{ a cover}]/2$

 $\mathbb{E}[w(C)|C \text{ a cover}] \le 2\mathbb{E}[w(C)] \le 2\ell \cdot LP = O(\log n) \cdot LP$

