Matchings: Max Cardinality and Min Cost

CSC 473 Advanced Algorithms
Matchings in graphs

• A matching in a graph \( G = (V, E) \) is a subset \( M \subseteq E \) of edges so that no two edges in \( M \) share an endpoint.

• **Maximum cardinality matching**: given input graph \( G \), find a matching \( M \) of maximum size
  • Perfect matching: size \( \frac{|V|}{2} \), i.e., all edges are matched
  • Solvable in polynomial time
Bipartite Graphs

- We will focus on max cardinality matching in bipartite graphs.
- **Bipartite graph**: $G = (V, E)$ so that we can partition $V$ into disjoint sets $A$ and $B$, and all edges in $E$ have one endpoint in $A$ and one in $B$
  - We can check if $G$ is bipartite in time $O(n + m)$
  - If it is, we can also find $A$ and $B$ in this time
- **Fact**: a graph is bipartite if and only if it does *not* have an odd cycle
  - *only if*: any path alternates $A$ and $B$ and can only come back to the starting node after an even numbers of hops.

Algorithm for general graphs is a deep result of Jack Edmonds.

Not bipartite
Bipartite Matching
Finding the Maximum Matching

- Greedy (keep adding edges while you can) does not work

- Do you know a polynomial time algorithm to find the max matching?
  - Compute a max flow
  - We will see a more combinatorial algorithm
Augmenting Paths

• For a bipartite graph $G$ and a matching $M$, a path $P$ is alternating if edges in $P$ alternate between being in $M$ and being outside of $M$.

• An alternating path is augmenting if it starts and ends in unmatched vertices.

1 -> a -> 2 -> b -> 3 -> c is alternating

2 -> b -> 3 -> c is augmenting
Augmenting Paths

- An alternating path is **augmenting** if it starts and ends in unmatched vertices.
- Let $P = \text{augmenting path}$. Set $M' = M \triangle P = (M \cup P) \setminus (M \cap P)$.
- $M'$ is a matching and $|M'| = |M| + 1$

2 -> b -> 3 -> c is augmenting

First and last vertex in $P$ unmatched, and others just switch which ones they are matched to.

One more non-matching edge in $P$ than matching edges: size of matching increases by 1.

Flip which edges are in $M$ along $P$. 
Characterizing Max Matchings

**Theorem.** A matching $M$ is of maximum cardinality if and only if there is no augmenting path for it.

- **only if:** Augmenting path means there is a larger matching
- **if:** Take a matching $M', |M'| > |M|$, and graph $H$ with edges $M \triangle M'$
  - $M \triangle M' = (M \cup M') \setminus (M \cap M')$
  - $H$ has max degree $\leq 2$
  - The connected components of $H$ are alternating paths and even cycles
  - $H$ has more edges from $M'$, so has a path component starting and ending with edges from $M'$: augmenting for $M$.
High-Level Algorithm

• $M = \emptyset$

• While $\exists$ an augmenting path $P$ for $M$
  • Set $M = M \triangle P$

• Correct by Theorem.

• At most $\frac{n}{2}$ iterations, since each iteration adds an edge to $M$

• How do we search for an augmenting path?
  • Will show a $O(n + m)$ time algorithm, for $O(n^2 + nm)$ total time
Finding Augmenting Paths

- Idea: construct a directed graph $G_M = (V, E_M)$
- alternating paths in $G \leftrightarrow$ directed paths in $G_M$
  - Direct edges in $M$ from right to left
  - Direct edges not in $M$ from left to right
  - Any path in $G_M$ must alternate edges in and outside $M$
  - Use BFS to search for a path in $G_M$ from an exposed vertex on the left to an exposed vertex on the right.
König’s Theorem

• Vertex Cover: a set $C$ of edges of $G = (V, E)$, so that for any edge $(a, b) \in E$, $a \in C$ or $b \in C$ (or both)

**Theorem.** In any bipartite graph, the size of the minimum cardinality vertex cover equals the size of the maximum cardinality matching.

• **Easy:** for any v.c. $C$ and matching $M$, $|M| \leq |C|
  • Edges in $M$ are disjoint, so no vertex in $C$ can cover more than one of them
  • True even for non-bipartite graphs.

• **Harder:** for the max matching $M$, there exists a v.c. $C$ s.t. $|C| = |M|
  • This part fails for some non-bipartite graphs.
Proof of Harder Direction

• $M = \text{max matching (no augmenting path)}$
• $U = \text{exposed vertices}$; $L = \text{vertices reachable in } G_M \text{ from } U \cap A$
• $C = (A \setminus L) \cup (B \cap L)$ is a vertex cover of size $|C| = |M|$

• No edges $(a, b)$ with $a \in A \cap L$ and $b \in B \setminus L$
  • $(a, b) \notin M$: if $a$ is reachable from $U$, then so is $b$
  • $(a, b) \in M$: $a \notin U$, and only incoming edge is $a \leftarrow b$, so $a$ is reachable from $U$ only if $b$ is

$L = \{d, e, 4\}$
$C = \{a, b, c, 4\}$
Proof, continued

- $M = \text{max matching (no augmenting path)}$
- $U = \text{exposed vertices; } L = \text{vertices reachable in } G_M \text{ from } U \cap A$
- $C = (A \setminus L) \cup (B \cap L)$ is a vertex cover of size $|C| = |M|$
  - Every vertex in $C$ touches exactly one edge in $M$
    - $A \setminus L \subseteq A \setminus U$ so all vertices in $A \setminus L$ are matched
    - All vertices in $B \cap L$ are matched, otherwise there is an augmenting path.
    - If $(a, b) \in M$, and $b \in B \cap L$, then $a \notin A \setminus L$ (if $b$ is reachable from $U$, so is $a$)

$L = \{d, e, 4\}$
$C = \{a, b, c, 4\}$