Random Assignment and Derandomization

CSC 473 Advanced Algorithms

Computer Science
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Maximum Satisfiability

- **Max SAT:** Given m clauses $C_1, \dots, C_m$ over variables $x_1, \dots, x_n$ find an assignment to the variables that maximizes the number of satisfied clauses
- Literal: a variable x_j or its negation $\overline{x_j}$
- A clause is a disjunction (OR) of (*any number of)* literals
  - E.g. $x_1 \vee \overline{x_5} \vee \overline{x_7} \vee x_{12}$, or x_{10}
  - Satisfied if at least one literal is true
- E.g. $C_1 = \overline{x_1}, C_2 = x_1 \vee \overline{x_2}, C_3 = x_1 \vee x_2 \vee x_3$.
  - $x_1 = F, x_2 = F, x_3 = T$ satisfies all clauses
- NP-hard to approximate better than a factor $\frac{7}{8}$. We will see factor $\frac{1}{2}$.

Random Assignment

- OPT = maximum number of satisfiable clauses
- *Upper bound* (trivial): $OPT \leq m$
  - At most every clause is satisfied
- **Lemma** Some assignment satisfies at least $\frac{m}{2} \geq \frac{OPT}{2}$ clauses.
- Set $x_j = B_j$, where $B_j \in \{T, F\}$, uniformly at random, independent
- $Y_i = 1 \Leftrightarrow C_i$ is satisfied. $Y = \sum_{i=1}^{m} Y_i$ is the number of sat. clauses
- If C_i has k literals, then $\mathbb{E}[Y_i] = \mathbb{P}(Y_i = 1) = 1 - 2^{-k} \geq \frac{1}{2}$

$$\mathbb{E}[Y] = \sum_{i=1}^{m} \mathbb{E}[Y_i] \geq \frac{m}{2}$$

Exactly one of 2^k choices of assignments to the literals in C_i makes it unsatisfied.

Deterministic Algorithm?

- Is there a deterministic (i.e., no randomness) algorithm that, for any clauses $C_1, \dots, C_m$ finds an assignment satisfying $\geq \frac{m}{2}$ clauses.
  - Achieves the approximation always, rather than in expectation

- *Idea*: set $x_1 = b_1, \dots, x_n = b_n$ one by one, so that

$$\mathbb{E}[Y | x_1 = b_1, \dots, x_j = b_j] \geq \mathbb{E}[Y] \geq \frac{m}{2},$$

holds for all j.

- $\mathbb{E}[Y | x_1 = b_1, \dots, x_n = b_n]$ is just the deterministic number of satisfied clauses.

Conditional Expectations

- How do we set x_1?
- Key observation:

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[Y|x_1 = T] \cdot \mathbb{P}(x_1 = T) + \mathbb{E}[Y|x_1 = F] \cdot \mathbb{P}(x_1 = F) \\ &= \mathbb{E}[Y|x_1 = T] \cdot \frac{1}{2} + \mathbb{E}[Y|x_1 = F] \cdot \frac{1}{2} \\ &\leq \max\{\mathbb{E}[Y|x_1 = T], \mathbb{E}[Y|x_1 = F]\}\end{aligned}$$

- Set $x_1 = T$ if $\mathbb{E}[Y|x_1 = T] \geq \mathbb{E}[Y|x_1 = F]$, and to F otherwise

Conditional Expectations

- More generally:

$$\mathbb{E}\left[Y|x_1 = b_1, \ldots, x_{j-1} = b_{j-1}\right]$$

$$= \mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = T\right] \cdot \frac{1}{2} + \mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = F\right] \cdot \frac{1}{2}$$

$$\leq \max\{\mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = T\right], \mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = F\right]\}$$

- Set $x_j = T$ if $\mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = T\right] \geq \mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = F\right]$, o/w to F

- By induction, $\mathbb{E}[Y|x_1 = b_1, \ldots, x_n = b_n] \geq \mathbb{E}[Y] \geq \frac{m}{2}$ for assignment $b_1, \ldots, b_n$

Computing the Conditional Expectations

- How do we compute $\mathbb{E}\left[Y|x_1 = b_1, \ldots, x_j = b_j\right]$?
- By linearity of expectation

$$\mathbb{E}[Y|x_1 = b_1, \ldots, x_i = b_i] = \sum_{i=1}^{m} \mathbb{E}\left[Y_i|x_1 = b_1, \ldots, x_j = b_j\right]$$

$$= \sum_{i=1}^{m} \mathbb{P}(Y_i = 1|\, x_1 = b_1, \ldots, x_j = b_j)$$

- Probability that C_i is satisfied conditional on $x_1 = b_1, \ldots, x_j = b_j$ is
  - 1, if one literal is already set to T
  - $1 - 2^{-k}$ if no literal set to T, and k literals still unset