Linear Algebra Review Sheet

1 Basic Definitions

An $m \times n$ matrix A over $\mathbb{R}$ is a two-dimensional table of real numbers, with rows indexed by $[m] = \{1, \ldots, m\}$ and columns indexed by $[n] = \{1, \ldots, n\}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

When $m = n$ we say that A is a square matrix. In a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal, often just called the diagonal. Sometimes we have the rows and columns of a matrix A be indexed by two arbitrary sets S and T, and we denote the entry indexed by $s \in S$ and $t \in T$ by a_{st}.

The transpose $B = A^\intercal$ of A is the $n \times m$ matrix given by $b_{ij} = a_{ji}$.

Special cases of matrices are the $n \times 1$ matrices, called column vectors, and $1 \times n$ matrices, called row vectors. Sometimes we will just say "vector" without specifying if it is a row or a column or row vector, when our discussion holds for either. We usually denote row and column vectors by lowercase letters. The space of all n-dimensional vectors is denoted by $\mathbb{R}^n$.

The product $C = AB$ of an $m \times n$ matrix A and an $n \times \ell$ matrix B is an $m \times \ell$ matrix defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Once again important special cases are when one of the matrices is a row or a column vector. In particular, if x is an m-dimensional row vector, and y an n-dimensional column vector, and A an $m \times n$ matrix, then

$$(xA)_j = \sum_{i=1}^{n} x_i a_{ij}, \qquad (Ay)_i = \sum_{j=1}^{m} a_{ij} y_j.$$

Above, xA is a row vector, and Ay is a column vector.

The matrix product in some ways behaves like the product of numbers: for any three matrices A, B, C we have $A(B + C) = AB + BC$ and $(A + B)C =$

$AC + BC$. However, it's important to note that the matrix product does not commute: in general AB does not have to equal BA. A special role is played by the identity matrix I which is the $n \times n$ matrix that has 1's on the diagonal and 0's everywhere else. For any $m \times n$ matrix A, $AI = A$, and for any $n \times m$ matrix B, $IB = B$.

An $n \times n$ matrix A has an inverse (i.e. is invertible) if there exists an $n \times n$ matrix B such that $AB = BA = I$, where I is the the $n \times n$ identity matrix. The inverse of A, if it exists, is unique and is denoted by A^{-1}. Not every matrix has an inverse.

2 Linear Subspaces and Linear Independence

A linear subspace W of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ such that:

1. For any $c \in \mathbb{R}$, and any $x \in W$, $cx \in W$.
2. For any $x, y \in W$, $x + y \in W$.

A subspace W is spanned by the vectors $x_1, \ldots, x_k$ if any vector $y \in W$ can be written as $y = c_1x_1 + \ldots + c_kx_k$ for some reals $c_1, \ldots, c_k$.

A set of vectors $\{x_1, \ldots, x_k\}$ in $\mathbb{R}^n$ is linearly independent if $c_1x_1 + \ldots c_kx_k = 0$ implies that $c_1 = \ldots = c_k = 0$. A basis of a subspace W is linearly independent set of vectors $\{x_1, \ldots, x_k\}$ that spans W. Every subspace W has a basis. All bases of W have the same cardinality, and we call this cardinality the dimension of W. In particular every basis of $\mathbb{R}^n$ has cardinality n. Every linearly independent set of vectors in W is a subset of some basis of W, i.e. can be completed to a basis.

Some important subspaces are associatd with an $m \times n$ matrix A. The column space of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A, and the rowspace of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A. The right nullspace of A is the set of column vectors $\{x \in \mathbb{R}^n : Ax = 0\}$, and the left nullspace of A is the set of row vectors $\{x \in \mathbb{R}^m : xA = 0\}$. When we say nullspace without specifying if it is the left or the right nullspace, we refer to the right nullspace.

3 Rank, Nullity, and Linear Systems of Equations

The rank of A, denoted rank A, equals the dimension of the column space of A. Equivalently, it also equals the dimension of the row space of A. It also equals the largest number of linearly independent columns of A, as well as the largest number of linearly independent rows of A.

The nullity of A, denoted nul A, equals the dimension of the right nullspace of A. Equivalently, it equals the cardinality of the largest set of linearly independent column vectors $x_1, \ldots, x_k$ such that $Ax_i = 0$.

The rank-nullity theorem says that for any $m \times n$ matrix

$$\text{rank } A + \text{nul } A = n.$$

Applying the theorem to the transpose of A we see that

$$\text{rank } A + \text{nul } A^{\mathsf{T}} = m.$$

Another fundamental result shows that an $n \times n$ matrix A has an inverse if and only if nul $A = 0$, which happens if and only if rank $A = n$.

Finally, we consider a system of linear equations

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

in the unknowns $x_1, \ldots, x_n$. This system can be compactly encoded as $Ax = b$. The following is a characterization of the solvability of such a system of equations:

1. If nul $A > 0$ (or, equivalently, rank $A < n$), and b is in the column space of A, then $Ax = b$ has infinitely many solutions.

2. If nul $A = 0$ (or, equivalently, rank $A = n$), and b is in the column space of A, then $Ax = b$ has exactly one solution.

3. If b is not in the column space of A, then $Ax = b$ has no solutions.

When $m = n$, and nul $A = 0$ (or, equivalently, rank $A = n$), the system $Ax = b$ has a unique solution for every b, and this solution is given by $x = A^{-1}b$.